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Missing Data Problem and the Empirical Yield Curve Analysis

An Example of T-bills Market in Armenia

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T-bills were introduced in 1995 and are still one of the most important instruments of the securities market of Armenia (85-95% of all operations of the secondary market in the last five years). The maturity yield of the state bonds during the last 7 years has been between 20% to 100%. Despite such a high proportion in the structure of the market, the low liquidity of the secondary market of state bonds does not allow building the yield curve with the help of standard methods and complicates the assessment of the interest rates in the economics. Authors build the yield curve based on the primary auctions which allows to more accurately assess the structure of the interest rates in the current situation. In this case the authors are solving a typical problem with missing data, for which they use two independent methods. One of the methods is based on space representation and the other one is based on sequencing correlated measures. Both methods are tested on generated data. Such approach may have methodological significance because it will allow comparison of the two methods of restoring missing data.

Keywords. Armenia, interest rate, missing data, Kalman filter, Kriging method.

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1. INTRODUCTION

There are two main goals in this study. The first one is based on the fact that one of the most important issues in understanding the overall situation in the financial market and for the adoption of state macroeconomic policy is determination of interest rates. The first goal of the project is the creation of a three-dimensional yield curve for the t-bills market of Armenia based on the data of primary auctions and testing a few hypotheses that are at the basis of theory of interest rates. The three-dimensional yield curve shows the dependence of the interest rate on the date of issuance and the period of maturity. This approach, where the data of the primary auctions are used for the creation of the yield curve, is justified by the low liquidity of the secondary market of government securities (GS) and by the absence of reasonably well organized trade space for conducting operations in the secondary market, where fair and transparent prices are formed. In the meantime, all the economic agents, the state structures and banks use the interest rate of GS for their everyday operations. Nonetheless, it is impossible to estimate the interest rates in the following cases:

1. For days when primary auctions are not held
2. For days when primary auctions are held with one issue only. Thus there is an interest rate only for one period of maturity. For other periods of maturity the information does not exist.

Thus the problem of creating the three-dimensional yield curve is a typical task of restoration of missing data. For this purpose most commonly Kalman Filter is used in econometrics. In this paper we suggest using Kriging Method in conjunction with Kalman Filter, which is a method not commonly used in econometrics, and therefore requires careful consideration. The latter is normally used in geostatistics for space variables. We use it in a phase space where time series is one of the variables. Thus, the second goal of this research is proposing a method of restoring missing data and describing unavailable financial-economic data with the help of Kriging Method and Kalman Filter, comparison of results received by different methods through the example of the T-bills market of Armenia and through generated data. The comparison of two independent methods of restoration of missing data - the Kalman Filter based on consecutive interdependent measurements and the Kriging method based on spatial presentation, is a central issue of this study.

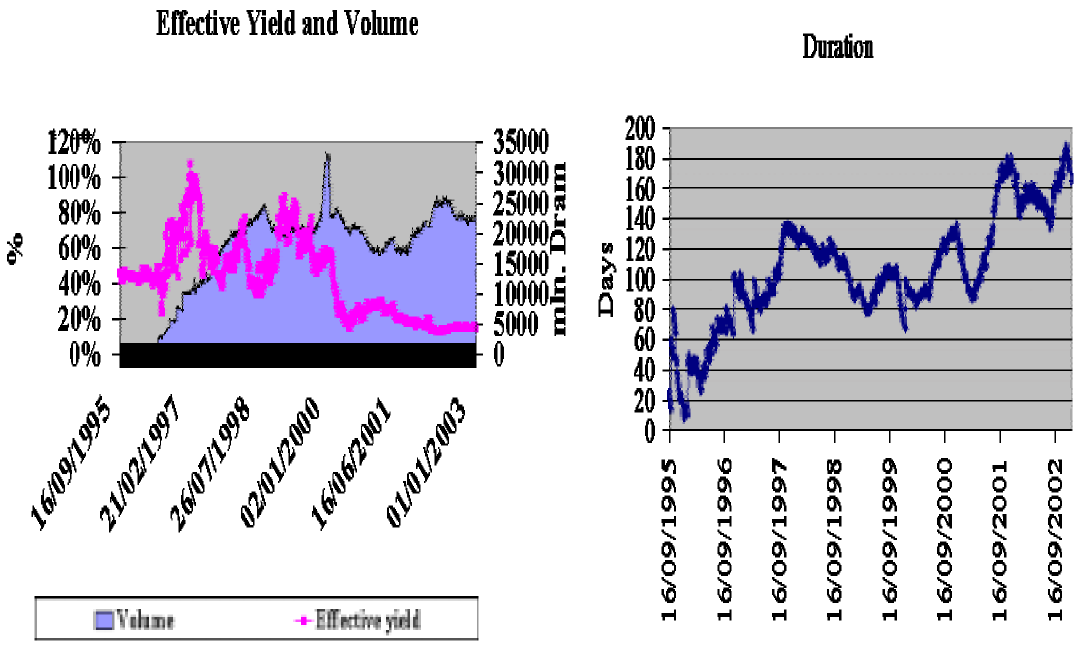
The problem of missing data is very typical of economies in transition. Therefore the use of adaptive methods may be useful not only for the creation of the yield curve, but may also be useful for other countries of the region, which face similar problems in their GS markets.

2. THE GOVERNMENT SECURITIES MARKET OF ARMENIA

The history of the Armenian government securities market starts from 1995. The first issues of T-bills were introduced that year. They were discounted securities with maturity periods of 28, 91, 182, 273 and 364 days. The issuing, maturity and circulation of T-bills resembled the Russian T-bills market. The emitter of T-bills was the Ministry of Finance of the Republic of Armenia. The primary issuing was done by the Central Bank (CB). The primary placement was done in auctions organized by CB. Banks and financial organizations participated in auctions, which have the status of dealer-banks or agent-banks. At different periods of the existence of T-bills market of Armenia the number of organizations allowed to participate in auctions varied, but practically all banks operating in Armenia had the right to participate.

In November 1999 amendments were made in the rules of placing T-bills. The securities could have maturity periods between 7 and 52 weeks, which expanded the range of possible securities issued. There were also changes connected with the nominal cost of T-bills: initially the nominal cost was 50000 AMD (about 100USD), but starting from 1999 investors could buy bonds with no less value than 10000 AMD with 1000 AMD increments. Besides, the concept of "treasury window" was introduced, which allowed investors to buy T-bills avoiding dealer-banks, at a weighted average price determined at an auction[1].

All these new approaches were aimed at activating the state securities markets, decreasing the average yield prior to maturity, increasing the duration of internal state debt, as well as minimizing negative effects of the Russian financial crisis in the T-bills market of Armenia. Before 1999 the state not always managed to place all the consecutive issues of T-bills, and this was true especially about 'long-term' issues. Starting from mid-2000 in almost all auctions the demand for T-bills exceeded the supply, which allowed to stabilize the level of interest rates and significantly increase the average duration of internal state debt.



Effective yield of auctions, T-bills volume in circulation and average duration of T-bills market of Armenia.

[1] At the same time the concept of agent-bank was introduced. Prior to that only dealer-banks existed. Agent-banks have more rights and responsibilities in placing state securities.

From March 2000 Public Treasury Coupon Mid-term bonds with partial maturity (PTCMB) were introduced. These are coupon bonds with circulation period of 1-5 years, which mature with equal parts (usually quarterly) during the period of circulation. However, the number of issues of such bonds during a year is limited. No more than 12 issues and no more than 2 issues a month can be emitted during a year.

As mentioned earlier, all these measures quite activated the primary T-bills market, however they did not have much effect on the secondary market. Operations on the secondary market happen sporadically. Besides, the absence of an organized trade space is even more important, which leads to the situation when information from the secondary market is not always accessible and reliable. The secondary market is practically the inter-bank market. In the one and only organized trade space of Armenia, the Armenian Stock Exchange, no single deal with T-bills was done in 2001-2002.

Currently the government securities sector is a pivotal link for the state monetary policy as well as a primary tool of government regulation of macroeconomic situation. At the same time, due to underdeveloped corporate securities market, GS act as key instruments of a newly-emerging capital market. For instance, about 90% of total stock exchange transactions done by professional market participants in 2001 referred to government securities trading. As the government securities market grew and expanded, the following issue acquired primary urgency: what role does this market play in economic reform? This problem is even more pressing under actual absence of a secondary market, since in this case the government policy of financial market has to rely exclusively upon primary auction results. In such conditions primary auctions are the only transparent enough market mechanism of forming interest rates. As a matter of fact, similar situation is observed in almost all transition countries.

3. LITERATURE

The literature review consists of two parts: 1) studies of restoring missing data and 2) studies of the time-structure of interest rates.

The problem of restoring the missing data occurs in many disciplines. However it has been most elaborated among natural sciences where problems of accuracy of measures, determination of values of non-measurable quantities based on measurements of dependent variables, and creation of algorithms for highly accurate equipment are common. Thus, it is not a coincidence that both methods of restoring missing data proposed in this project were initially created for use in natural sciences. Kalman Filter was designed for use in technical devices analyzing signals (Kalman, 1960). There is a vast amount of literature on use of Kalman Filter in technical equipment. An extensive review of such research may be found in the book by Jacobs (1993). The use of Kalman Filter in econometrics for the analysis of time series is most comprehensively described in the work of Hamilton (1994). This Filter is used in econometrics for different aspects of analysis of time series. The application of the Kalman Filter specifically for the problem of restoring missing data can be found in the works by Harvey and Pierce (1984) and Kohn and Ansley (1986), in which functions of maximum likelihood for ARIMA models with missing data are discussed. Modification of Kalman Filter and the algorithm of restoring missing data for the case of infinite variation functions is presented in the publication by De Djong (1991).

The Kriging Method is most frequently used in geostatistics. The detailed description of the method is provided in the book of Davies (1986). The ordinary Kriging Method and the trans-Gaussian Method, which we use here are described in works of Ripley (1987) and Gressie (1993) respectively.

It is noteworthy that recently the two methods are being used jointly (e.g., Mardia et al., 1998 and Cressie and Wikle, 2002).

The problem of missing data often occurs in economics too. This problem is not just inherent of transitional economies. Cuche and Hess (1999) discuss the issue of estimating the monthly value of GDP of Switzerland on the basis of quarterly data during the period of 1981-1997 with the help of Kalman Filter. The methodology of such estimation was proposed by Harvey (1989).

In the foreign literature there are many works dedicated to the analysis of yield curves and related strategies of the market. There are five main hypotheses explaining the time-structure of interest rates:

1. Expectations Theory, according to which the expected excess yield has a constant value which is the same for bonds with all maturity periods (Mishkin 1997). There is a so-called Pure Expectations Theory, which states that long-term interest rates are equal to the average short-term interest rate (Campbell, Lo, McKinley, 1997).

2. Liquidity Preference Theory, according to which the forward rate premium for the period is constant in time but depends on the period of maturity (Woodward, 1983).

3. Market Segmentation Theory is based on the assumption that different investors may have different preferences in regard with periods of investment (Culbertson, 1957).

4. Time Varying Risk Premium, which takes into consideration the possibility of the effect of external variables of condition on the level and sign of the forward rate premium during the period (Balduzzi, Bertola, Foresi, 1997).

5. Preferred Habitat Theory assumes that the investor has his own horizon of investments and prefers to buy bonds with maturity periods not exceeding it. The time structure of interest rates is determined by the joint response to the independent actions of all the investors (Mankiw, Miron, 1986). Note that this hypothesis may be viewed as a variety of the Market Segmentation Hypothesis.

A more detailed analysis of yield curves and of models explaining the time structure of interest rates is done in works of Cochrane (2000) and Drobishevsky (1999).

Peculiarities of building the yield curve and testing the hypotheses explaining the time-structure of interest rates for transition economies has been very rudimentarily studied, and mostly through the case of T-bills market of Russia. Among these works the work of Entov et. al. (1998) tests the expectations hypothesis based on weekly and monthly data during the period of January 1994-January 1998. The results do not allow to reject the hypothesis of rational expectations for the Russian market, but the hypothesis cannot be accepted either. The expectations of the market participants expressed through T-bills forward rate are biased in relation to the future spot rate, and at the same time forward rates carry part of the information about the level of future yield of T-bills.

4. HOW TO BUILD THE YIELD CURVE?

Some authors, such as Drobishevsky (1999) call the three-dimensional yield curve "the dynamics of the yield curve". This title rather accurately describes the existing methodology of building a three-dimensional yield curve. It is built in the following way. At first the time period is chosen for building the curve. Then for each point of chosen period a usual (two-dimensional) yield curve is built. The resulting two-dimensional yield curves are combined in one three-dimensional chart. Thus, at first for each point in time two-dimensional curves are built, and then three-dimensional curves are built based on that.

Building of yield curves even for the most developed markets, e.g. for T-bills markets of the US, is also associated with the problem of missing data, since for each moment of time not always dealings with corresponding maturity periods are available (e. g. Campbell, Lo, C. McKinley, 1997). In order to solve the problem of missing data in building the yield curve, usually the method of spline functions is used, which was proposed by McCulloch (1971, 1975). In this approach the researcher has to make a choice between the accuracy of approximation (goodness-to-fit) and smoothness of analytical curves (e.g. van Deventer and Imai, 1997).

It is natural, that the problem of missing data during building of yield curves is more urgent for economies in transition with less liquid securities markets. Studies of the interest rate structures based on the data from T-bills market of Russia faced this problem in one way or another. In publication Entov et. al. (1998) the method of approximation through linear spline functions is used. In work by Drobishevsky (1999) an averaging method was used – that of linear interpolation of prices for groups of securities with close maturity periods. In the work by Kryukovskaya (2003) approximation through cubic spline functions was used.

In this project we propose a different approach for building the yield curve. First we build the three-dimensional yield curve. Since building the yield curve is almost always associated with the problem of missing data, building the three-dimensional yield curve during approximation allows to simultaneously take into consideration not only associations among interest rates of various maturity periods for a specific point in time, but also associations among interest rates for different points in time. In case of building a two-dimensional curve, these associations cannot be taken into consideration during approximation.

The smaller is the amount of initial data the more important becomes using all associations during approximation. In other words, the less liquid is the market, the more important it is to use all degrees of freedom during approximation. Apparently, after building the three-dimensional yield curve it is very easy to receive the two-dimensional curve for any moment of time by section of the curve with the corresponding plane. Thus, the proposed method is very useful for such low-liquidity markets as the government securities market of Armenia. However, it can be used for even more developed markets where commonly the problem of missing data exists too. Moreover, a comparative analysis of curves received through traditional methods and through the method proposed here may be done using data from such markets. It is important to mention that the traditionally used method for approximation of two-dimensional yield curves – spline functions, as shown by Wahba (1990) and Mardia et. al. (1996) is a particular case of Kriging Method. Thus, using Kriging Method for approximation of yield curves may be of great importance as a possibility of development of the method of spline functions.

5. DESCRIPTION OF THE DATA

For this research primary T-bills auctions data is used. It is important to note that at the end of 1999 significant changes were made in the order of issuing T-bills: up to November of 1999 issues were with maturity periods of 28, 91, 182, 273, 364 days. Starting from November 1999, according to the new rules, T-bills with 1-52 week maturity periods may be issued, which makes the solution of our problem more precise.

This data is fully authentic. Information on primary auctions is appropriated by the CB of Armenia. Since there is no organized secondary market in Armenia, no official data is available. Meanwhile, some secondary market data may also be taken from the CB for several reasons: first, CB is an active market participant, second, all other market participants are accountable to CB. However, the absence of the organized trade space leads to the situation where the information from the secondary market is not always available and reliable, and most importantly is not applicable for creation of the yield curve. Thus, the CB provides information from the secondary market on a weekly basis in the format presented in Table 1.

Table 1.

	Up to 3 months	Up to 6 months	Up to 9 months	Up to 12 months
Average yield Volume of transactions				

The table provides the weighted average yield of operations conducted during a given week. Note that transactions with a wide range of maturity periods (3 months) are being averaged, and the specific day of transaction is not being mentioned. These make the information practically useless for building the yield curve on a daily basis. In addition, by using the data of the secondary market the problem of missing data still remains due to the low liquidity of the market. However in that case the missing data will not have an orderly structure. The data of the primary auctions are orderly because the auctions are regularly held on Tuesdays and Thursdays. It is well known that working with orderly data is more convenient (e. g. Harvey, 1989).

In this research the yield curve is built for the period of January 2000-December 2001. Attachment 1 presents the checked database of T-bills primary auctions for the study period. The attachment represents only three parameters used in the study: issue date, maturity period and yield. Initially it was planned to study a longer period, starting from the first auction up to present day. However, it turned out that a rather powerful computer is unable to perform the calculations of Kriging Method for such a big database. This fact does not pose a significant limitation because it does not interfere with the methodology, and the study period remains long enough, so that the results may to used for further research of the market.

The study period consists of 495 days (excluding weekends). During that period 168 auctions were held on placing T-bills with different periods of maturity. The maximum period of maturity was 364 days. Thus, our task is to determine the interest rate for each point of the two-dimensional phase space where the X-axis is the date and the Y-axis is the maturity period. To solve this problem we intend to use the Kriging Method and the Kalman Filter. The applicability of these methods largely depends on the stationarity of the data, therefore we first conducted the augmented Dickey-Fuller test for unit roots. Figure 1 presents yields to maturity of T-bills formed at the primary auctions.

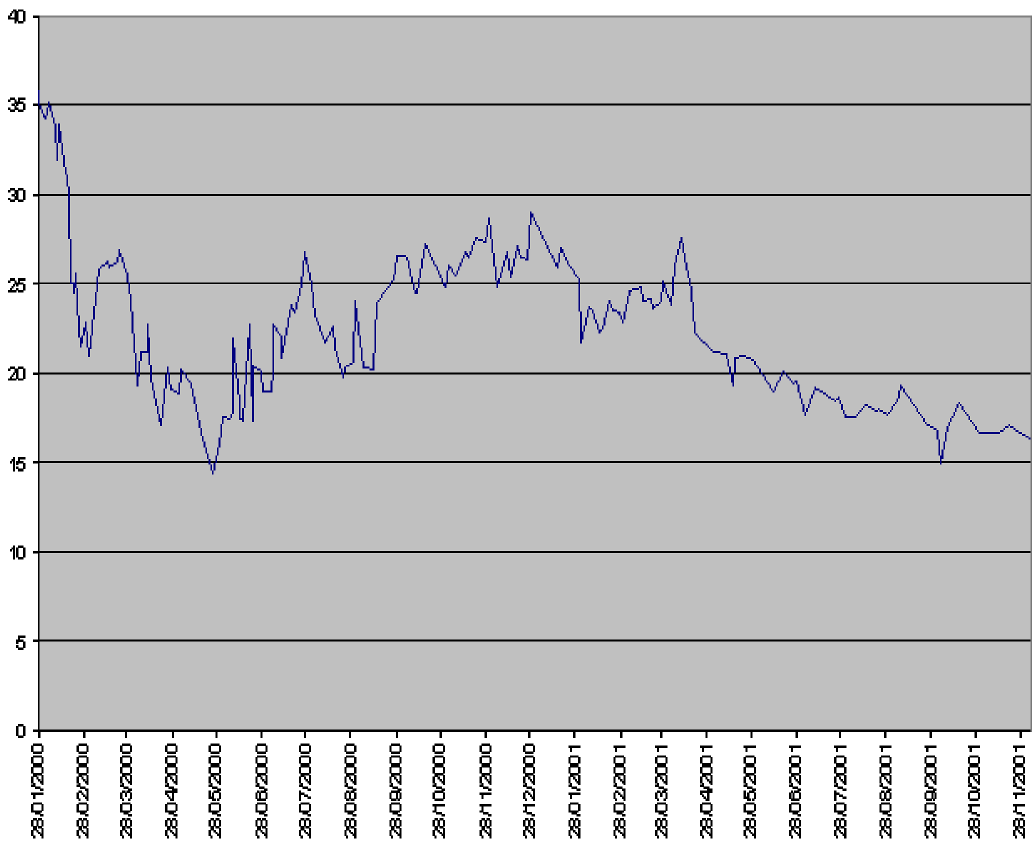


Fig. 1. Yields to maturity of T-bills of Armenia.

Table 2 presents results of the test with one lag in E-views format.

Table 2. Augmented Dickey-Fuller test for unit root.

ADF Test Statistic	-2.672344	1% Critical Value*	-2.5780
		5% Critical Value	-1.9417
		10% Critical Value	-1.6167

*MacKinnon critical values for rejection of hypothesis of a unit root.

Sample(adjusted): 3 168

Included observations: 166 after adjusting endpoints

Variable	Coefficient	Std. Error	t-Statistic	Prob.
X(-1)	-0.015338	0.005739	-2.672344	0.0083
D(X(-1))	-0.272598	0.073878	-3.689861	0.0003

R-squared	0.093985	Mean dependent var	-0.186747
Adjusted R-squared	0.088460	S.D. dependent var	1.860614
S.E. of regression	1.776413	Akaike info criterion	3.999045
Sum squared resid	517.5257	Schwarz criterion	4.036539
Log likelihood	-329.9207	Durbin-Watson stat	2.032207

As seen from the table we have to reject the null hypothesis about non-stationarity of the process for 1% level of significance. This result allows using both Kriging Method and Kalman Filter. Figure 2 shows yields to maturity of primary auctions of T-bills with the trend in 3-D field. This graph is a 3-D yield curve with missing data. It will serve as a basis for building the continuous 3-D yield curve.

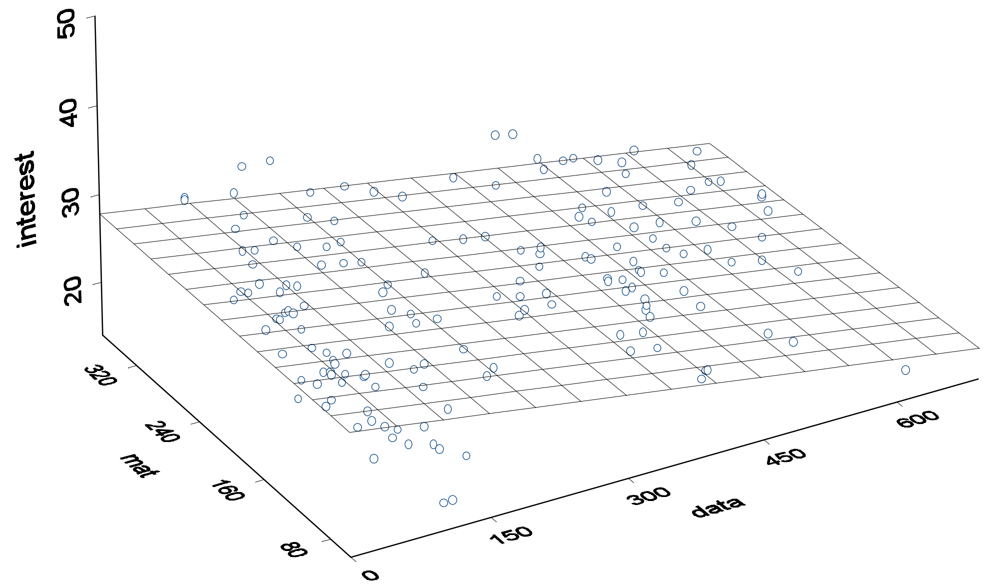


Fig. 2. 3-D yield curve with missing data.

6. METHODOLOGY

6.1. The Kriging Method

We will consider two ways to do the prediction: ordinary Kriging, which is the most popular method (Ripley, 1987), and the trans-Gaussian model (Gressie, 1993, Kozintseva, 1999).

6.2. Ordinary Kriging Method

Kriging Method (or ordinary Kriging Method) consists of consideration of an unknown function *Z(s)* as realization of some random function in a space of changes of *s* in a way that this data becomes involved in this realization. Within it estimation of *Z(s)* should be consistent with BLUE (the Best Linear Unbiased Estimator). In our case Z(s) is the yield function defined on the two-dimensional space the axes of which are the dates of placement of the T-bills and the maturity periods. A random function *Z(s)* which complies with

$$E(Z(s_i)) = \mu$$

$$Cov(Z(s_i), Z(s_j) = C(s_i - s_j)$$

for all s_i and s_j is called a second-order stationary. Function $C(s_i - s_j)$ is called covariogram, or a function of stationary covariation.

A correlation function (normalized covariation function) applied within this method is stationary as well as isotropic since it depends only on a space between the points and not on the direction.

As far as the value of a correlation function in the point of 0 always equals 1, and since no variation may equal 1, then an additional parameter called 'precision of a random value' is introduced. Within it parametric dependence between the variation and a correlation function is the following:

$$Cov(Z(s_i), Z(s_j) = C(s_i - s_j) = \frac{1}{\tau} r_\theta \left(s_i - s_j\right),$$

consequently,

$$Cov(Z(s_i), Z(s_j) = C(0) = Var(Z) = \frac{1}{\tau} r_\theta (0) = \frac{1}{\tau}$$

where $r_\theta (s_i - s_j)$ is a correlation function while θ is an independent parameter. $r_\theta (l)$ indicates isotropic correlation function with θ parameter and l space between the points in which covariation is calculated.

Since the types of associations among the points of the yield curve are not known a priori, we intend to use four correlation functions.

- Exponential correlation function

$$r_\theta = \theta_1^{l^{\theta_2}},$$

where $\theta_1 \in (0,1)$ and $\theta_2 \in (0,2)$,

- a function of rational square correlation

$$r_\theta(l) = \left(1 + \frac{l^2}{\theta_1^2}\right)^{-\theta_2},$$

where $\theta_1 > 0$ and $\theta_2 > 0$,

- a function of Matern correlation

$$r_\theta(l) = \begin{cases} \dfrac{1}{2^{\theta_2 - 1}\Gamma(\theta_2)}\left(\dfrac{l}{\theta_1}\right)^{\theta_2} K_{\theta_2}\left(\dfrac{l}{\theta_1}\right), & l \neq 0 \\ 1 & l = 0 \end{cases},$$

where $\theta_1 > 0$ and $\theta_2 > 0$, while K_{θ_2} is a modified Bessel third-order function of θ_2,

- a function of spherical correlation

$$r_\theta(l) = \begin{cases} 1 - \dfrac{3}{2}\left(\dfrac{l}{\theta}\right) + \dfrac{1}{2}\left(\dfrac{l}{\theta}\right)^3, & l \le 0 \\ 0 & l > 0 \end{cases},$$

where $\theta > 0$.

Attachment 2 contains a detailed description of derivation of the algorithm of the method of ordinary Kriging and the method of trans-Gaussian Kriging.

The value of the yield function $Z(\vec{s}_0)$ in an arbitrary point s_0, is defined in the following way, based on the observations $Z(\vec{s}_1) \ldots Z(\vec{s}_n)$ and the selected correlation function C:

- Set vector $\vec{Z} = (Z(s_1),\ldots,Z(s_n))^T$
- Set vector $\vec{c} = (C(s_0 - s_1),\ldots,C(s_0 - s_n))^T$
- Set tensor $C_{ij} = C(s_i - s_j)$
- Calculate $m = \dfrac{1 - \sum_i (\vec{C}^{-1}\vec{c})_i}{\sum_i \sum_j (\vec{C}^{-1})_{ij}}$
- Calculate vector $\vec{\lambda} = \vec{C}^{-1}(\vec{c} + \vec{m})$, where $\vec{m} = (m, m, \ldots, m)^T_{1xn}$
- Calculate the variance $\sigma_k^2(s_0) = C(0) - \vec{\lambda}^T \vec{c} + m$
- The value of $Z(\vec{s}_0)$ in point $\vec{s}_0$ is defined in the following way $\hat{p}(\vec{Z}, \vec{s}_0) = \vec{\lambda}^T \vec{Z}$.

6.3. Trans-Gaussian Kriging Method

Another option of the Kriging Method which we would like to use is the trans-Gaussian Kriging Method.

Suppose now the $Z(\vec{s})$ process is obtained from

$$Z(\vec{s}) = \phi(Y(\vec{s}))$$

where $Y(\vec{s})$ is stationary Gaussian process and ϕ is one-to-one twice-differentiable function. The idea is to transform problem from Z scale to the Y scale, predict $Y(\vec{s}_0)$ and transform the result back. The problem with this approach is that it gives biased predictor (see Attachment 2).

The family of transformations that we discuss here is the Box-Cox family: for x>0

$$g_\lambda(x) = \begin{cases} \dfrac{x^\lambda - 1}{\lambda}, for \lambda \neq 0 \\ \log x, for \lambda = 0 \end{cases}$$

Note that the reverse Box-Cox transformation may lead to significant increase of variance. For example, when Y is normally distributed with parameters N(5,1), then for λ=1, Z ~N(6;1), for λ=0.5, Z ~N(12.5;12.375) and for λ=0, Z ~N(244.69;102880.6).

Therefore, the algorithm for the trans-Gaussian Kriging is as follows.

The values $Z(\vec{s}_1) \ldots Z(\vec{s}_n)$ at points $\vec{s}_1 \ldots \vec{s}_n$, the covariance function C, the transformation ϕ which transforms $\vec{Z}(\vec{s}\ ) = \phi(\vec{Y}(\vec{s}))$ for a Gausian field Y, and the location of interest $\vec{s}_0$ are given:

- Set database $\vec{Z} = (Z(s_1),...,Z(s_n))^T$.
- Construct a new database $\vec{Y}$ as transformation ϕ of database $\vec{Z}$.
- Calculate $\vec{Y}(\vec{s}_0)$ through ordinary Kriging for base $\vec{Y}$.
- Do reverse transformation $\breve{p}_Z(\vec{Z},\vec{s}_0) = \phi(\hat{p}_Y(\vec{Y},\vec{s}_0)) + \phi''(\hat{\mu}_Y)(\dfrac{\sigma_Y^2(\vec{s}_0)}{2} - m_Y)$.

6.4. Cross-validation

For defining the best curve we will conduct a cross validation procedure for generated and real data. That is, throw away 20 data points at a time and predict them using the remaining 148 points for real data. Compare the mean squared errors. While doing the cross validation we also calculate the 95% prediction intervals and compare their average lengths and the proportion of points that lie outside their prediction intervals.

We use the following procedure for the comparison.

1. Generate a Gaussian random field Y on a 50 by 50 grid. We use Gaussian fields with mean 3, variance 1, and will every time specify the correlation function.

2. Sample 30 random points from Y.

3. Take an inverse Box-Cox transformation of Y with fixed parameter λ. This gives us the field of interest Z, and in particular the inverse Box-Cox transformation of the 30 sampled points are the given data Z.

4. Perform the prediction of all 2500 points of the grid given the 30 data points. Compare the mean squared error (this is possible only for the generated data).

5. Perform cross validation. That is, throw away one data point at a time and predict it using the remaining 29 points. Compare the mean squared errors (this is possible for both the generated and real data). While doing the cross validation we also calculate the 95% prediction intervals and compare their average lengths and the proportion of points that lie outside their prediction intervals.

6.5. The Kalman Filter

The condition-space representation is given by a system of two vector equations. First, the condition or transition equation describes the dynamics of the condition vector (ξ_t) containing the unobserved variables we want to estimate. The second type of equation represents the observation or measurement equation linking the condition vector to the vector containing the observed variables (y_t). The equations of this system for t = 1, … , T where T is the number of observations, are the following:

$$\vec{\xi}_{t+1} = F_t\vec{\xi}_t + C_t\vec{x}_{t+1} + R_t\vec{v}_{t+1},\ \vec{y}_t = A_t\vec{x}_t^* + H_t\vec{\xi}_{t+1} + N_t\vec{w}_t .$$

In addition to the unobserved and the observed variables of interest, vector equations contain the so-called related series ($\vec{x}_t$) and ($\vec{x}_t^*$) as exogenous variables in each equation. Both equations have error terms multinormally distributed:

$$\begin{pmatrix} v_t \\ w_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & G \end{pmatrix}\right).$$

The coefficients matrices $F_t, C_t, R_t, A_t, H_t, N_t$ and the two variance-covariance matrices Q and G are estimated by maximizing the log-likelihood function of this system. The detailed description of the algorithm of the Kalman Filter can be found in the works by Hamilton (1994), where necessary transformations are described for cases with non-normal data. In our case vector $\vec{y}_t$ has dimension of 168x1 and consists of the observed points of the yield curve, vector $\vec{\xi}_t$ has dimension of 364x1, and consists of all points of the yield curve. $\vec{x}_t^*$ is a vector with dimension 168x1, and its components are the respective periods of the observed points. Vector $\vec{x}_t$ has the following components (495, 0, …, 0). The procedure of Kalman Filter is used 495 times, and only the components of vector $\vec{x}_t$ are changed each time. The second time the Kalman Filter will be used with the components of vector $\vec{x}_t = (494,0,...0)$, etc. The final application will be with vector $\vec{x}_t = (1,0,...0)$. As a result we will have 495 values for vector $\vec{\xi}_t$ for each day of investigated period.

The best result received through Krging Method and through Kalman Filter will be studied through the cross validation procedure for real and generated data described earlier for the selection of the best yield curve from the group of yield curves received through Kriging Method. Thus we will select the best curve, which will best correspond to the yield curve of the T-bills market of Armenia.

7. RESULTS

7.1. Generated data

Generation of a random Gaussian space with an average value of zero and with the given function of correlation r on the space of coordinates $\vec{S} = \{s_{ij}\}$ with dimensions of n_1xn_2, means generating n_1xn_2 jointly normal random variables. Usually these variables are gathered in row-wise vector $\vec{Z}$ starting from the upper left corner of the space of coordinates. The space of coordinates together with the correlation function determines the covariation matrix $\vec{C}$. Thus, our task is generation of vector $\vec{Z} \sim N(\vec{0}_{n1n2}, \vec{C}_{n1n2xn1n2})$. Usually, Cholesky Method is used for this. However here we will use the so called Circulant Embedding Method (Dietrich and Newsam,1993, Kozintsev, 1999), the description of which is brought in Attachment 3.

Based on this algorithm we conduct the following experiment. Figures 3, 4, and 5 show results of applying the Kriging algorithm. First we generated a Gaussian random field (Figure 3), than sampled 30 random points from it, and used them as data for restoring missing data and cross-validation procedure. Such experiment is usually performed in technical disciplines (see for example Kozintseva, 1999).

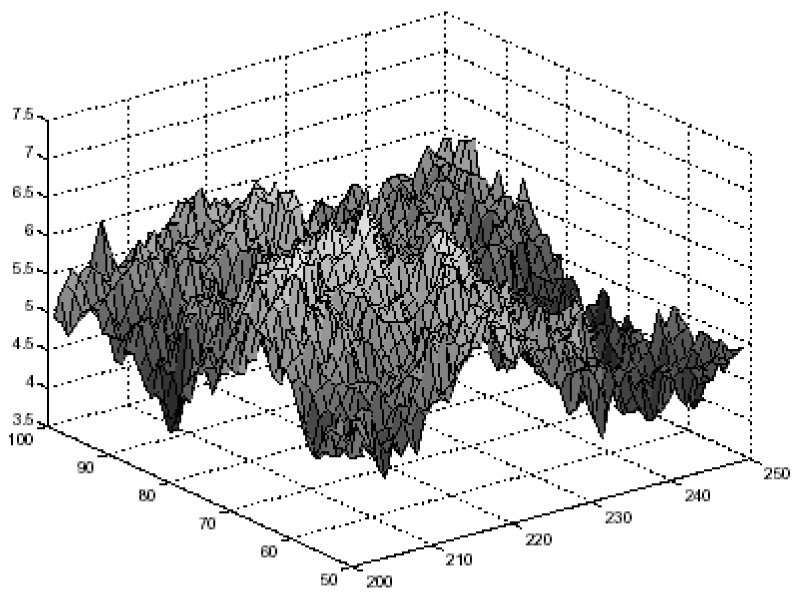


Fig. 3. Gaussian(3, 1) random field.

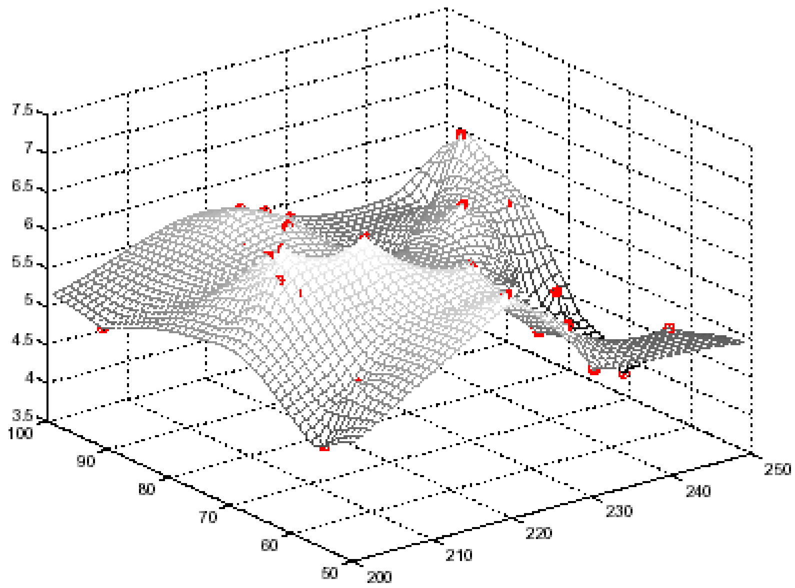


Fig. 4. Kriging surface obtained from 30 data points, exponential(exp(-0.02), 1) correlation.

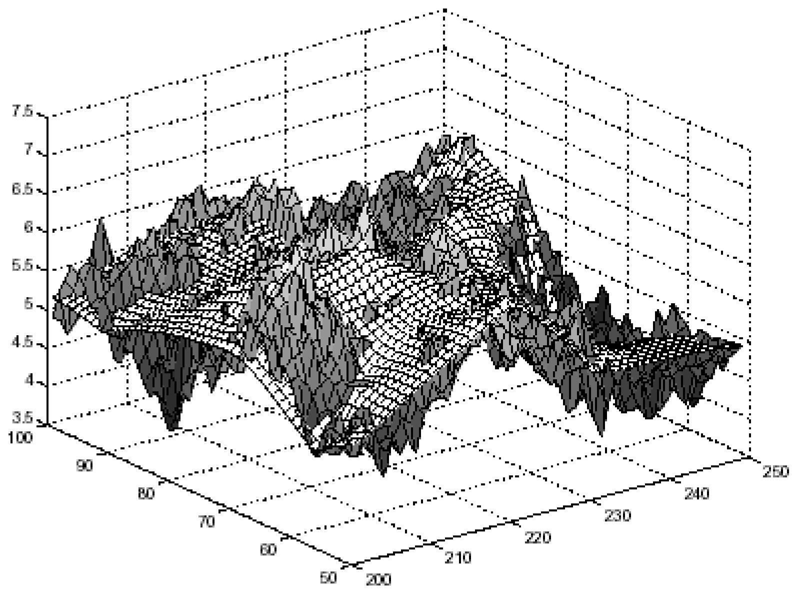


Fig. 5. Overlay of a Gaussian random field and its Kriging approximation from 30 data points.

Figure 4 shows the resulting Kriging surface. Figure 5 shows the original Gaussian field overlaid with its Kriging prediction. In all the figures $\tau = 1$

Below are the results received for generated data.

Tables 3 and 4 show the values of MSE of predicting all 2500 points.

The big error for $\lambda = 0$ (lognormal data) in two methods is due to the big variance in the field. As one can see in the tables, in some cases the Kalman Filter yields better data than the Kriging

Method (for all cases with $\lambda = 0.5$, except for exponential and spherical correlation functions, and for $\lambda = 0$, except for exponential correlation function), but is always worse in comparison with ordinary Kriging Method when $\lambda = 1$. At the same time the Trans-Gaussian Kriging Method yields close results with the Kriging Method. Thus, only by the value of MSE for all generated points it is impossible to give preference to any of the methods.

Table 3. Mean squared error for the Exponential and Matern correlations.

Correlation function	Exponential ($e^{-0.03}$,1)			Matern (1,10)		
Transformation parametr	λ=0	λ=0.5	λ=1	λ=0	λ=0.5	λ=1
KRIG	9645.788	3.901	0.543	59123.112	6.439	0.825
TGK	9834.736	4.298	0.543	59814.712	6.987	0.825

Table 4. Mean squared error for the Rational Quadratic, Spherical correlations and Kalman Filter.

Correlation function	Rational Quadratic (0.9,1)			Spherical (50)		
Transformation parametr	λ=0	λ=0.5	λ=1	λ=0	λ=0.5	λ=1
KRIG	115654.865	13.980	1.765	55235.809	5.157	0.752
TGK	116679.108	13.102	1.765	54780.217	5.452	0.752

	Kalman		
	λ=0	λ=0.5	λ=1
MSE	53874.430	5.963	3.236

Now we perform cross validation for generated data. Tables 5 and 6 show the mean squared errors, the average lengths of the 95% prediction intervals (denoted by l), and the percentages of the points that are outside of their prediction interval (denoted by `out').

Table 5. Cross Validation on 50 data points: exponential and Matern correlations. The entries are MSE, average length of the 95 % PI, and the percentage of the observations outside their PI.

Correlation function		Exponential ($e^{-0.03}$,1)			Matern (1,10)		
Transformation parameter		λ=0	λ=0.5	λ=1	λ=0	λ=0.5	λ=1
KRIG	MSE	13456.134	2.323	0.117	56098.754	8.081	0.439
	Out	93%	60%	3%	98%	57%	7%
	L	1.218	1.324	1.555	3.003	2.5128	2.014
TGK	MSE	12372.430	1.742	0.117	54111.607	6.873	0.439
	Out	20%	7%	3%	23%	10%	7%
	L	231.127	5.785	1.555	246.486	8.866	2.014

Table 6. Cross Validation on 50 data points: spherical, rational quadratic correlations and Kalman Filter. The entries are MSE, average length of the 95 % PI, and the percentage of the observations outside their PI.

Correlation function		Spherical (50)			Rational Quadratic (0.9,1)		
Transformation parametr		λ=0	λ=0.5	λ=1	λ=0	λ=0.5	λ=1
KRIG	MSE	2756.342	1.512	0.130	29173.097	8.342	1.782
	Out	100%	60%	3%	93%	57%	7%
	L	71.730	3.328	1.298	93.675	11.873	3.666
TGK	MSE	2556.452	1.443	0.130	28973.332	9.653	1.874
	Out	17%	10%	3%	7%	7%	3%
	L	165.320	4.901	1.298	552.24	13.590	3.569

	Kalman		
	λ=0	λ=0.5	λ=1
MSE	23456.325	5943.327	4223.667
"out"	100%	80%	77%
L	279.321	8.218	7.869

From these tables we can see that there is a big difference in the accuracy of the prediction intervals. Due to the difference in the variance of the fields all three methods have better accuracy of the prediction intervals for the fields closer to Gaussian. The ordinary Kriging PI's are based on the Gaussian assumption, because of this PI's for $\lambda = 0$ and $\lambda = 0.5$ are unrealistically narrow and miss most of the true values.

The same situation exists in the case of Kalman Filter which does not contain the majority of base data in the PI either. At the same time, the closer the base space to Gaussian, the more base points are contained in the prediction interval. At the same time, the ordinary Kriging Method yields significantly better results than the Kalman Filter when $\lambda = 0$. In general, for values of interest of λ the best results are obtained when using spherical and exponential correlation functions.

Up to this point we used the exact values for all the parameters in ordinary and trans-Gaussian (TG) Kriging. In practice we do not have them. To study the effects of misspecifying the parameters we now perform ordinary and trans-Gaussian Kriging with some of the parameters fixed at values close to the true values, but not exactly equal.

Tables 7 and 8 show the results of the trans-Gaussian Kriging performed with different values of the transformation parameter λ deviating from the true λ^* for spherical correlation function (50).

Table 7. TG Kriging with different values of λ. $\lambda^* = 1$ is the true parameter value.

	λ=1.5	λ=1.2	λ^*=1	λ=0.8	λ=0.5	λ=0
MSE	0.135	0.129	0.130	0.133	0.134	0.145
Out	53%	20%	3%	3%	0%	0%
L	0.985	1.126	1.298	1.978	2.443	6.225

Table 8. TG Kriging with different values of λ. $\lambda = 0$ is the true parameter value.

	λ=1	λ=0.8	λ=0.5	λ=0.3	λ^*=0	λ=-0.1
MSE	2843.219	2696.153	2598.563	2601.754	2556.452	2790.197
Out	87%	90%	83%	77%	17%	10%
L	13.486	50.325	103.664	146.124	165.320	322.920

The pattern here is that for smaller values of λ the length of the prediction intervals increases. So when λ is overestimated we have too small prediction intervals, and when λ is underestimated, the MSE becomes bigger and the length of prediction intervals increases very fast. In other words, the length of the PI's increases with λ. Next we keep the true value of λ fixed but change the correlation parameter.

The correct value of MSE is practically minimal in both cases. At the same time the number of points outside of the prediction interval is not minimal for correct values of λ. However those values of λ, when the number of points outside of the prediction interval is smaller than for the correct value of λ, have unrealistically wide prediction intervals. Thus, for the correct parameter λ rather wide prediction interval and MSE value close to the minimal is typical.

Table 9 shows the results for the spherical (50) correlation for various fixed values of λ. Cross validation by the trans-Gaussian Kriging was performed with the true value of λ and different values of the correlation parameter.

For all three choices of λ the minimum of the MSE was achieved for the true parameter value θ^*= 50. The length of PI's decreases as θ increases and as a result the number of points not lying within the confidence interval increases.

Next we will conduct an experiment with different numbers of base points. The table below presents results of experiment for exponential ($e^{-0.2}$,1) and spherical (50) correlation functions by trans-Gaussian Kriging method, as well as for Kalman Filter with different numbers of N for base points. For all functions parameter λ=0.5.

Table 9. TG Kriging with various values of the correlation parameter. $\theta^* = 50$ is the true parameter value.

		θ = 10	θ = 25	θ = 40	θ^* = 50	θ = 75	θ = 100	θ =125
λ=0	MSE	25786.125	9498.342	3487.125	2556.452	5339.754	7903.651	9984.978
	Out	7%	7%	13%	17%	27%	47%	63%
	L	763.981	532.087	274.130	165.320	125.238	65.543	49.461
λ=0.5	MSE	15.347	4.532	1.532	1.443	2.981	4.948	9.403
	Out	7%	7%	13%	10%	17%	23%	30%
	L	11.662	8.297	2.337	1.298	1.147	1.006	9.136
λ=1	MSE	1.143	0.264	0.184	0.130	0.154	0.179	0.435
	Out	3%	0%	0%	3%	3%	10%	20%
	L	1.975	1.534	1.321	1.298	1.145	1.076	0.906

Table 10. Results of the Experiment with Different Values of Base Points.

N		30	100	500	1000	1500	2000
Exponetial	MSE for 2500 points	4.298	3.564	2.966	2.901	2.565	2.605
	MSE for N points	1.742	1.632	1.297	1.255	1.165	1.129
	Out	7%	6%	4%	5%	3%	5%
	L	5.785	5.032	4.391	3.938	3.361	3.045
Spherical	MSE for 2500 points	5.452	5.239	4.424	3.025	2.698	2.156
	MSE for N points	1.443	1.335	1.016	1.093	0.989	1.034
	Out	10%	8%	8%	7%	8%	5%
	L	4.901	4.605	3.591	3.603	3.015	2.988
Kalman	MSE for 2500 points	5.963	5.803	5.021	4.852	3.896	2.457
	MSE for N points	5943.327	4021.782	1903.406	612.003	31.341	6.453
	Out	80%	57%	46%	31%	11%	8%
	L	8.218	7.342	5.089	5.214	4.325	3.175

It could be expected that the more base points are taken, the better results are obtained for all three methods. As seen from the table, it is especially true about Kalman Filter. Some deviation from the rule for parameter “out” is explained by narrowing of the prediction interval.

The most important result received in the experiment with different numbers of base points is that when N is changed, the shape of the curve remains unchanged.

7.2. Real data

Based on real data 15 curves were built through the Kriging Method based on three correlation functions: exponential, spherical and rational quadratic with different parameter values. Attachment 4 presents all 15 curves together with the curve built through the Kalman Filter.

Below are the cross-validation procedure results for real data. The cross validation procedure for real data was performed the following way: we left out 20 points (Attachment 5 represents coordinates of these points, real values, predicted values and the prediction interval) and then we predict them with the help of the remaining 148 data points.

Table 11.

	Rational Quadratic (θ_1,θ_2, τ, λ)				
	(10, 3, 1, 0)	(2, 2, 1, 0.5)	(20, 3, 1, 1)	(25, 4, 1, 0.5)	(25, 2, 1, 1)
Var	8.7509	9.8090	5.8424	5.8046	4.3888
"out"	75.00%	5.00%	65.00%	0.00%	55.00%
L	3.9075	19.0100	3.5640	16.6831	2.9045

Table 12.

	Exponential (θ_1,θ_2, τ, λ)				
	(0.1, 0.1, 1, 0)	(0.5, 0.5, 1, 0.5)	(0.9, 0.9, 1, 0.5)	(0.9, 0.9, 1, 1)	(0.99, 0.99, 1, 0)
Var	8.8630	9.4886	5.2048	5.1806	2.1951
"out"	75.00%	5.00%	0.00%	65.00%	65.00%
L	3.9186	18.9668	17.6644	3.7009	1.6406

Table 13.

	Spherical (θ,τ, λ)				
	(0.9, 1, 1)	(300, 1, 0)	(300, 1, 0.5)	(300, 1, 1)	(180, 180, 1)
Var	9.9707	2.1040	2.0436	2.1040	2.0655
"out"	80.00%	85.00%	15.00%	85.00%	65.00%
L	3.9332	1.1765	5.8942	1.1765	1.5173

Table 14.

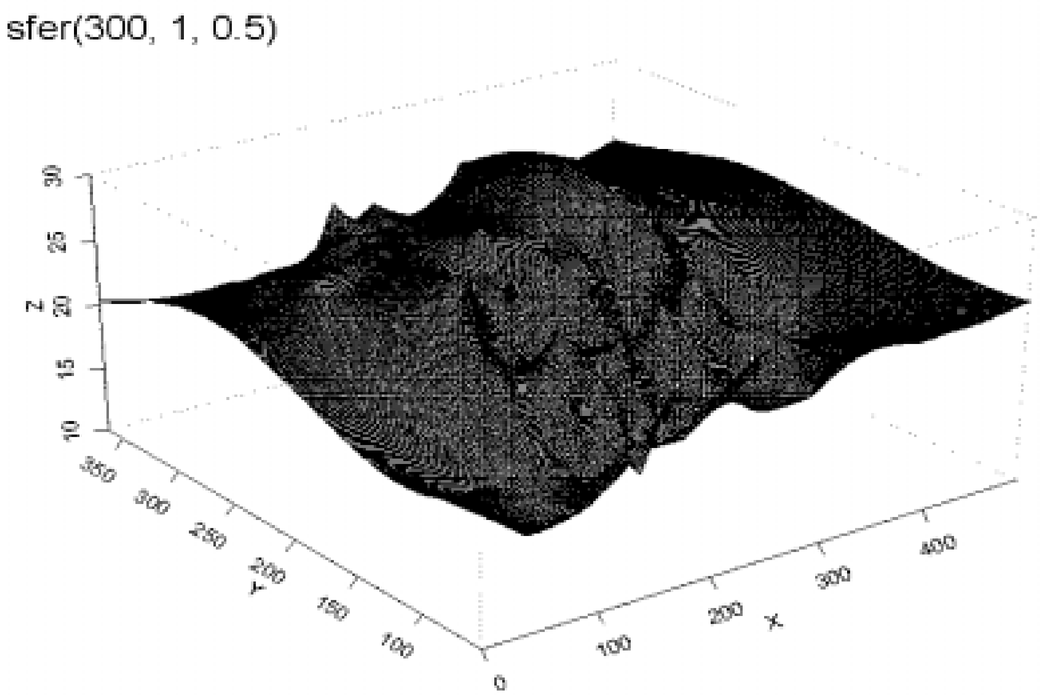
	Kalman
Var	9.5296
"out"	75.00%
L	3.9304

The table shows that the smallest value of variance is obtained when the curve is built by Transgaussian Kriging Method (λ=0.5) through spherical quadratic function with parameter θ=300. The experiment with generated data showed that when the value of parameter θ is close to real, the value of MSE is the smallest. All three curves received with the help of spherical correlation function with parameter θ=300 have a rather small value of MSE that allows to expect that parameter θ=300 is close to real. This curve has also the best indicator for the number of points not included in the prediction interval among all curves built through spherical correlation function. However, better values for this indicator have the curves built by Transgaussian Kriging Method (λ=0.5) through:

1. Exponential function (θ_1=0.9,θ_2=0.9).
2. Rational quadratic (θ_1=25,θ_2=4).
3. Exponential function (θ_1=0.5,θ_2=0.5).
4. Rational quadratic (θ_1=2,θ_2=2).

The first two curves give a zero result for "out", i.e. all predicted values are included in the predicting interval, while the two others produce a 5% result. Besides, the first two curves have the smallest value of parameter MSE for the families of exponential and rational quadratic correlation functions respectively. However all four curves have very wide predicting intervals which explain the better results for “out”.

The experiment with generated data showed that in case of incorrect specification predicting intervals increase unrealistically, and thus the number of points outside of the predicting intervals decreases. However all four curves have very wide confidence intervals which explain the better results for “out”. The curves built through Transgaussian Kriging Method (λ=0.5) with spherical correlation function (θ=300) have the best indicators: for the lowest variance value and a rather narrow predicting interval the “out” values are 15%. Note that the best results were received when Transgaussian Kriging was used (λ=0.5).



Use of the Kalman Filter, just like in the case with generated data, produced rather bad results in case of a rather wide predicting interval: ¾ of the base points occurred outside it. In this case the value of the mean square error is one of the largest among the received curves.

8. POSSIBILITIES FOR MODEL IMPROVEMENT

8.1. Kriging Method

The further development of Kriging Method may be in the following directions:

1. Determination of covariance function based on observed data.

2. Determination of the parameters of covariance function (θ_1, θ_2) and parameter λ for Trnas Gaussian Kriging through the method of maximum likelihood.

8.2. Kalman Filter

Obtained results show that application of Kalman Filter produces worse estimation in comparison with Kriging Method. This result is in contradiction with the few works in which these two methods were compared. These studies were conducted mainly in the areas of geostatistics and electronic engineering (e. g. Todini, 2001). Indeed, application of Kriging Method assumes choice of correlation function (rather arbitrary), while Kalman Filter parameters are calculated with quite high precision. In our case the problem is the following: application of Kalman Filter assumes that the components of the condition vector do not contain missing data, and restoration of the missing data is done only for the time component. In our case the condition vector also contains missing data. Therefore, we have to use the Kalman Filter twice. First we restore the missing data of the condition vector (this turns out to be impossible since condition vector components also depend on time – maturity periods), and only after that restore the yield curve. That is why for our purposes it would be more correct to consider the Kalman Filter in the following way (we may call it a two-dimensional filter):

$$\vec{\xi}_{t+1,\tau+1} = F_{t,\tau}\vec{\xi}_{t+1,\tau} + G_{t,\tau}\vec{\xi}_{t,\tau+1} + Q_{t,\tau}\vec{\xi}_{1,\tau} + C_{t,\tau}\vec{x}_{t+1,\tau+1} + R_{t,\tau}\vec{v}_{t+1,\tau+1},$$

$$\vec{y}_{t,\tau} = A_{t,\tau}\vec{x}^{*}_{t,\tau} + H_{t,\tau}\vec{\xi}_{t+1,\tau+1} + N_{t,\tau}\vec{w}_{t,\tau}.$$

8.3. Kriged Kalman Filter

Another direction of further research is application of abovementioned method of Kriged Kalman Filter, the essence of which is the following: let

$$x(s,t) = \mu(s,t) + \varepsilon(s,t)$$

where μ is the mean value, and ε is error.

Under specific conditions the mean may be presented in the following way:

$$\mu(\vec{s},t) = h_1(\vec{s})\alpha_1(t) + h_1(\vec{s})\alpha_1(t) + \ldots h_1(\vec{s})\alpha_1(t) = \vec{h}(\vec{s})^T\vec{\alpha}(t)$$

Substitution of this equation into the previous one for each vector $\vec{s}$ produces observation equation, and the condition equation may be written in the following way:

$$\vec{\alpha}(t) = P\vec{\alpha}(t-1) + K\vec{\eta}(t)$$

where $\vec{\eta}(t)$ ~NID(0, Σ)

$\varepsilon(\vec{s},t)$ is presented in the form of correlation process:

$$\operatorname{cov}(\varepsilon(\vec{s},t),\varepsilon(\vec{s}',t')) = 0 \text{ for } t \neq t', \text{ all s } s'$$

The last four equations are called Kriged Kalman Filter (Mardia et. el., 1998).

9. TESTING OF PURE EXPECTATIONS HYPOTHESIS

The tree-dimensional yield curve is a convenient instrument for empirical testing of the expectations theory, which is a classical theory for understanding the form of the yield curve. The applicability of the expectations theory gives the opportunity to make predictions about the expected interest rates.

The most common method of testing the expectations hypothesis is estimation of linear regression equations. We tested the applicability of expectations theory for the T-bills market of Armenia based on specification of the model proposed on works of Campbell (Campbell, 1995, Campbel et el, 1997), which exists in two forms. The first one states that the expected yield of one-period bond must be equal to the yield of multi-period bond sold after one period. In mathematical form this statement is expressed in the following way:

$$(1+Z_{1,t}) = (1+Z_{\tau,t})^{\tau} E_t[(1+Z_{\tau-1,t+1})^{-(\tau-1)}]$$

where $Z_{\tau,t}$ is the yield to maturity τ on day t.

The second form of expectations hypothesis is formulated in the following way: expected yields of one-period and n-period securities invested for n periods must be equal to:

$$= (1+Z_{\tau,t})^{\tau} = (1+Z_{1,t})E_t[(1+Z_{\tau-1,t+1})^{\tau-1}]$$

In the general case the two forms of testing the expectations theory do not coincide because

$$E_t[(1+Z_{\tau-1,t+1})^{-(\tau-1)}] = E_t[(1+Z_{\tau-1,t+1})^{\tau-1}].$$

Thus, based on the first form of testing the expectations theory the following econometric model can be constructed:

$$z_{\tau-1,t+1} - z_{\tau,t} = \alpha_1(\tau) + \beta_1(\tau)(z_{\tau,t} - z_{1,t})/(\tau-1) + \varepsilon(\tau)$$

where we introduced logarithmic yields

$$z_{\tau,t} = \ln(1+Z_{\tau,t}).$$

The econometric model based on the second form of testing the expectations theory is expressed in the following way:

$$\sum_{i=1}^{\tau-1}(z_{1,t+i}/(\tau-1) - Z_{1,t} = \alpha_2(\tau) + \beta_2(\tau)((\tau-1)/\tau)(Z_{\tau,t} - Z_{1,t}) + \varepsilon(\tau)$$

If the expectations theory is fulfilled then coefficients β_1 and β_2 must be statistically significant and close to one.

During estimation the method of least squares with Newey-West estimation of covariance matrix was used. The results of the preformed test on the whole interval of interest are presented in the following table:

Table 15.

Dependent variable	Maturity (weeks)				
	1	2	6	15	30
β_1	-1.543**	-0.438***	0.673***	1.066**	1.523*
β_2	2.345*	0.237*	0.189***	0.207***	0.466**

*** - significacant at 99%, **- significacant at 95%, * - significacant at 90%.

As seen from the table, we received results confirming the expectations theory, as well as contradicting with it. Only the first form of the hypothesis of pure expectations is confirmed for the time horizon of 15 weeks. Coefficient β_2 for all investment periods is close to zero except for the one-week investment horizon where its value is close to 2. Thus, the second form of the hypothesis is rejected for all investment periods under study, which again confirms the fact that state security markets of countries with transition economies are oriented toward short-term securities.

The received results are quite close to the results received in works (Kryukovskaya, 2003), where only the first form of testing hypothesis of pure expectations is confirmed. However the investment horizon is somewhat smaller – 6 weeks, which can be explained by the study interval (April 1996 – August 1998), when investors were targeting more rapid profits.

Testing hypothesis of pure expectations gives controversial results even in the case of developed markets. However we would like to mention two peculiarities of transition economies.

- Short time-series of observations and short investment horizon. For overcoming this problem daily and weekly time periods are necessary to study, which leads to strengthening of noise associated with short-term fluctuations.
- The second peculiarity is more important from the point of view of our research – it is the existence of missing data and necessity of approximation of the yield curve. Apparently the method of approximation and the number of missing data affect the results of hypothesis testing. However, usually the approximation leads to smoothing of data but does not change the form of the yield curve which allows to receive results quite close to real.

10. CONCLUSION

Three-dimensional yield curve is built based on two methods: Kriging Method and Kalman Filter for the T-bills market of Armenia. The curve with the best characteristics is chosen by means of the cross-validation procedure among the family of curves built through the methods mentioned above. For the Armenian T-bills market such curve turned out to be the curve built by means of trans-Gaussian method with spherical correlation function (300).

The conducted experiment with generated data allowed comparing the two methods of building the curve. It turned out that when the number of base points, that serve for building the curve, is small the best results are yielded through curves built either by Kriging Method or by trans-Gaussian Kriging Method. When the number of base points is bigger, the Kalman Filter yields results that are comparable with those obtained through the Kriging Method. When the number of base points is increased, all methods yield better results, however results for Kalman Filter are improved at a higher rate.

The most important conclusion of the experiment with generated data is that increasing the number of base points does not lead to changing the shape of the curve, but only to the smoothing of it. This conclusion allows using three-dimensional yield curve received based on real data for econometric research such as:

1. construction of a model of inter-relation of the T-bills market and other segments of the financial market based on the analysis of one-day yields of the T-bills market, as well as estimation of the level of integration in the world financial markets (Ivanter, Persetski,1999);
2. construction of a macroeconomic model that includes one-day yields of the T-bills market (Gurvich, Dvorkovich,1999).

Additionally the curve provides the opportunity for the government agencies responsible for market regulations to effectively make policies of placing and managing GS portfolio (this method is already in use by the Securities Commission of Armenia for estimation of interest rates).

This project tested the hypothesis of pure expectations theory for T-bills market of Armenia based on the built curve. It turned out that the second form of the hypothesis stating that the expected yields of one-period and n-period securities invested for n periods must be equal, does not hold for the period under consideration in the T-bills market of Armenia. At the same time the first form of the hypothesis of expectations theory stating that the expected yield of one-period bond must be equal to the yield of multi-period bond sold after one period, holds for Armenian T-bills market for investment horizon of 15 weeks. These results are rather close to results obtained for the T-bills market of Russia, and most likely this is an indication of similarities of GS markets of countries with transitional economies which warrants taking into consideration segmentation and liquidity of the market for explanation of the shape of the yield curve.

APPENDIX

Attachment 1

No	Data	Maturity	Interest
1	11-Jan-00	147	50
2	13-Jan-00	105	47
3	18-Jan-00	189	46
4	20-Jan-00	126	42
5	21-Jan-00	203	42
6	25-Jan-00	196	40
7	26-Jan-00	133	37
8	27-Jan-00	147	36
9	28-Jan-00	273	36
10	28-Jan-00	189	35
11	01-Feb-00	161	34
12	03-Feb-00	280	35
13	08-Feb-00	140	34
14	09-Feb-00	154	32
15	10-Feb-00	217	34
16	15-Feb-00	77	31
17	16-Feb-00	168	31
18	17-Feb-00	154	30
19	18-Feb-00	119	25
20	21-Feb-00	182	25
21	21-Feb-00	140	25
22	22-Feb-00	203	26
23	24-Feb-00	133	23
24	25-Feb-00	168	21
25	29-Feb-00	168	23
26	02-Mar-00	84	21
27	09-Mar-00	252	26
28	14-Mar-00	210	26
29	16-Mar-00	133	26
30	21-Mar-00	161	26
31	23-Mar-00	245	27
32	28-Mar-00	161	25
33	30-Mar-00	189	24
34	04-Apr-00	140	19
35	06-Apr-00	294	21
36	11-Apr-00	189	21
37	11-Apr-00	175	23
38	13-Apr-00	133	20
39	20-Apr-00	56	17
40	25-Apr-00	196	20
41	25-Apr-00	182	20
42	27-Apr-00	154	19
43	27-Apr-00	133	19
44	02-May-00	203	19
45	04-May-00	308	20
46	11-May-00	280	19
47	16-May-00	126	17

No	Data	Maturity	Interest
48	18-May-00	147	16
49	25-May-00	84	14
50	30-May-00	154	16
51	01-Jun-00	294	18
52	06-Jun-00	119	17
53	06-Jun-00	112	17
54	08-Jun-00	203	18
55	08-Jun-00	364	22
56	13-Jun-00	231	18
57	13-Jun-00	196	17
58	15-Jun-00	140	17
59	20-Jun-00	329	23
60	22-Jun-00	98	17
61	22-Jun-00	189	20
62	27-Jun-00	154	20
63	29-Jun-00	364	19
64	04-Jul-00	133	19
65	06-Jul-00	175	20
66	06-Jul-00	287	23
67	11-Jul-00	210	22
68	11-Jul-00	168	21
69	18-Jul-00	273	24
70	20-Jul-00	217	23
71	25-Jul-00	259	25
72	27-Jul-00	294	27
73	01-Aug-00	175	25
74	03-Aug-00	119	23
75	10-Aug-00	217	22
76	15-Aug-00	308	23
77	17-Aug-00	259	21
78	22-Aug-00	224	20
79	24-Aug-00	336	20
80	29-Aug-00	266	21
81	31-Aug-00	364	24
82	05-Sep-00	182	20
83	12-Sep-00	154	20
84	14-Sep-00	238	24
85	26-Sep-00	350	25
86	28-Sep-00	315	27
87	03-Oct-00	140	27
88	05-Oct-00	252	26
89	10-Oct-00	154	25
90	12-Oct-00	182	25
91	17-Oct-00	301	27
92	31-Oct-00	175	25
93	02-Nov-00	245	26
94	07-Nov-00	182	26
95	09-Nov-00	147	26
96	14-Nov-00	231	27
97	16-Nov-00	161	26
98	21-Nov-00	196	28
99	28-Nov-00	182	27
100	30-Nov-00	287	29
101	05-Dec-00	84	25
102	12-Dec-00	196	27

No	Data	Maturity	Interest
103	14-Dec-00	105	25
104	19-Dec-00	203	27
105	21-Dec-00	70	26
106	26-Dec-00	91	26
107	28-Dec-00	266	29
108	16-Jan-01	105	26
109	18-Jan-01	175	27
110	23-Jan-01	140	26
111	25-Jan-01	189	26
112	30-Jan-01	168	25
113	01-Feb-01	63	22
114	06-Feb-01	175	24
115	08-Feb-01	133	24
116	13-Feb-01	70	22
117	15-Feb-01	70	22
118	20-Feb-01	161	24
119	22-Feb-01	147	23
120	27-Feb-01	154	23
121	01-Mar-01	182	23
122	06-Mar-01	168	25
123	13-Mar-01	245	25
124	15-Mar-01	175	24
125	20-Mar-01	189	24
126	22-Mar-01	210	24
127	27-Mar-01	245	24
128	29-Mar-01	364	25
129	03-Apr-01	168	24
130	05-Apr-01	350	26
131	10-Apr-01	294	27
132	12-Apr-01	175	26
133	17-Apr-01	217	25
134	19-Apr-01	161	22
135	02-May-01	294	21
136	11-May-01	350	21
137	15-May-01	168	19
138	17-May-01	364	21
139	22-May-01	280	21
140	29-May-01	238	21
141	12-Jun-01	119	19
142	19-Jun-01	315	20
143	26-Jun-01	231	19
144	28-Jun-01	364	20
145	03-Jul-01	112	18
146	10-Jul-01	273	19
147	24-Jul-01	308	18
148	26-Jul-01	364	19
149	31-Jul-01	238	18
150	07-Aug-01	217	18
151	14-Aug-01	350	18
152	21-Aug-01	273	18
153	23-Aug-01	364	18
154	28-Aug-01	301	18
155	04-Sep-01	210	18
156	06-Sep-01	364	19
157	11-Sep-01	301	19

No	Data	Maturity	Interest
158	25-Sep-01	266	17
159	02-Oct-01	196	17
160	04-Oct-01	77	15
161	09-Oct-01	245	17
162	16-Oct-01	315	18
163	30-Oct-01	350	17
164	06-Nov-01	322	17
165	13-Nov-01	273	17
166	20-Nov-01	364	17
167	27-Nov-01	294	17
168	04-Dec-01	301	16

Attachment 2

Ordinary Kriging

The ordinary Kriging predictor satisfies the following two assumptions.

1. Model Assumption:

the field Z is second order stationary with the unknown mean μ,

$$Z(\vec{s}) = \mu + \delta(\vec{s}), s \in D, \mu \in R, \tag{2.1}$$

where σ is a zero-mean, second-order stationary process with covariogram $C(\vec{h}), h \in R^2$.

2. Predictor Assumption: the predictor $p(\vec{Z}, \vec{s}_0)$ is linear and satisfies

$$p(\vec{Z}, \vec{s}_0) = \sum_{i=1}^{n} \lambda_i Z(\vec{s}_i), \text{ где } \sum_{i=1}^{n} \lambda_i = 1. \tag{2.2}$$

The condition that the coefficients of the linear predictor sum to 1 guarantees uniform unbiasedness:

$$E(p(\vec{Z}, \vec{s}_0)) = E\sum_{i=1}^{n} \lambda_i Z(\vec{s}_i) = E(\vec{Z}(\vec{s}_0)\sum_{i=1}^{n} \lambda_i = \mu$$

Of all the predictors satisfying the model and prediction assumptions, the ordinary Kriging predictor is defined as an optimal one, where the word "optimal" refers to the squared-error loss. Therefore the goal is to minimize the mean-squared prediction error

$$\sigma_e^2(\vec{s}_0) \equiv E(Z(\vec{s}_0) - p(\vec{Z}, \vec{s}_0))^2$$

with respect to the predictor coefficients.

To find the ordinary kriging predictor we need to minimize the function f,

$$f(\lambda_{1,} ..., \lambda_{n,} m) = E(Z(\vec{s}_0) - P(\vec{Z}, \vec{s}_0))^2 - 2m\sum_{i=1}^{n} \lambda_i - 1, \tag{2.3}$$

with respect to λ_1, ..., λ_n and m (the parameter m is a Lagrange multiplier). For our model (2.1) and predictor (2.2) the expression (2.3) becomes:

$$
\begin{aligned}
f(\lambda_{1,} ..., \lambda_{n,} m) &= E(Z(\vec{s}_0) - \sum_{i=1}^{n} \lambda_i Z(\vec{s}_i))^2 - 2m\left(\sum_{i=1}^{n} \lambda_i - 1\right) = \\
&= E(Z(\vec{s}_0)^2) - 2E\left(Z(\vec{s}_0)\sum_{i=1}^{n} \lambda_i Z(\vec{s}_i)\right) + E\left(\left(\sum_{i=1}^{n} \lambda_i Z(\vec{s}_i)\right)^2\right) - 2m\left(\sum_{i=1}^{n} \lambda_i - 1\right) = \\
&= C(\vec{0}) + \mu^2 - 2\sum_{i=1}^{n} \lambda_i E(Z(\vec{s}_0)Z(\vec{s}_i)) + \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j E(Z(\vec{s}_i)Z(\vec{s}_j)) - 2m\left(\sum_{i=1}^{n} \lambda_i - 1\right) = \\
&= C(\vec{0}) + \mu^2 - 2\sum_{i=1}^{n} \lambda_i (C(\vec{s}_0 - \vec{s}_i) + \mu^2) + \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j (C(\vec{s}_i - \vec{s}_j) + \mu^2) - 2m\left(\sum_{i=1}^{n} \lambda_i - 1\right) = \\
&= C(\vec{0}) - 2\sum_{i=1}^{n} \lambda_i C(\vec{s}_0 - \vec{s}_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j C(\vec{s}_i - \vec{s}_j) - 2m\left(\sum_{i=1}^{n} \lambda_i - 1\right) = \\
&= C(\vec{0}) - 2\sum_{i=1}^{n} \lambda_i C(\vec{s}_0 - \vec{s}_i) + \sum_{i=j} \lambda_i \lambda_j C(\vec{s}_i - \vec{s}_j) + \sum_{i \neq j} \lambda_i \lambda_j C(\vec{s}_i - \vec{s}_j) - 2m\left(\sum_{i=1}^{n} \lambda_i - 1\right) = \\
&= C(\vec{0}) - 2\sum_{i=1}^{n} \lambda_i C(\vec{s}_0 - \vec{s}_i) + 2\sum_{i<j} \lambda_i \lambda_j C(\vec{s}_i - \vec{s}_j) + C(\vec{0})\sum_{i=1}^{n} \lambda_i^{\;2} - 2m\left(\sum_{i=1}^{n} \lambda_i - 1\right).
\end{aligned}
\tag{2.4}
$$

Differentiating, we get

$$\frac{\partial f}{\partial \lambda_1} = 2\lambda_1 C(\vec{0}) + 2\sum_{1<j} \lambda_j C(\vec{s}_1 - \vec{s}_j) - 2C(\vec{s}_0 - \vec{s}_1) - 2m = 2\sum_{j=1} \lambda_j C(\vec{s}_1 - \vec{s}_j) - 2C(\vec{s}_0 - \vec{s}_1) - 2m,$$

.

.

$$\frac{\partial f}{\partial \lambda_n} = 2\lambda_n C(\vec{0}) + 2\sum_{j \neq n} \lambda_j C(\vec{s}_n - \vec{s}_j) - 2C(\vec{s}_0 - \vec{s}_n) - 2m = 2\sum_{j=1} \lambda_j C(\vec{s}_n - \vec{s}_j) - 2C(\vec{s}_0 - \vec{s}_n) - 2m,$$

$$\frac{\partial f}{\partial m} = -2\left(\sum_{i=1}^{n} \lambda_i - 1\right)$$

Now we need to solve the linear system:

$$\sum_{j=1} \lambda_j C(\vec{s}_1 - \vec{s}_j) = C(\vec{s}_0 - \vec{s}_1) + m,$$

.

.

$$\sum_{j=1} \lambda_j C(\vec{s}_n - \vec{s}_j) = 2C(\vec{s}_0 - \vec{s}_n) + 2m,$$

$$\sum_{i=1}^{n} \lambda_i = 1.$$

Denote

$$\vec{1} = (1, 1, ..., 1)^T_{1xn},$$
$$\vec{c} = (C(\vec{s}_0 - \vec{s}_1), ..., C(\vec{s}_0 - \vec{s}_n))^T,$$
$$\vec{C} \to C_{ij} = C(\vec{s}_i - \vec{s}_j), i, j = 1, ..., n,$$
$$\vec{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n),$$
$$\vec{m} = m\vec{1}.$$

Then we can rewrite the system as

$$\vec{C}\vec{\lambda} = \vec{c} + \vec{m}, \sum_{i=1}^{n} \lambda_i = 1. \tag{2.5}$$

The first equation gives $\vec{\lambda} = \vec{C}^{-1}(\vec{c} + \vec{m})$.

Plugging this into the second equation we get:

$$\sum_{i=1}^{n} \lambda_i = \vec{1}^T \vec{\lambda} = \vec{1}^T \vec{C}^{-1}(\vec{c} + \vec{m}) = \vec{1}^T \vec{C}^{-1}\vec{c} + \vec{1}^T \vec{C}^{-1}\vec{m} = 1.$$

Therefore,

$$\vec{m} = \frac{1 - \vec{1}^T \vec{C}^{-1}\vec{c}}{\vec{1}^T \vec{C}^{-1}\vec{1}} = \frac{1 - \sum_i \left(\vec{C}^{-1}\vec{c}\right)_i}{\sum_i \sum_j (\vec{C}^{-1})_{ij}}. \tag{2.6}$$

So the λ that minimizes *f* is given by

$$\hat{\vec{\lambda}} = \vec{C}^{-1}\left(\vec{c} + \frac{1 - \sum_i \left(\vec{C}^{-1}\vec{c}\right)_i}{\sum_i \sum_j (\vec{C}^{-1})_{ij}}\vec{1}\right). \tag{2.7}$$

and the ordinary kriging predictor is $\hat{p}(\vec{Z}, \vec{s}_0) = \hat{\vec{\lambda}}\vec{Z}$.

To simplify the notation we write λ for $\hat{\vec{\lambda}}$. The minimized mean-squared prediction error is called the kriging variance and is denoted by $\sigma_k^2(\vec{s}_0)$. For the ordinary kriging predictor $\hat{p}(\vec{Z}, \vec{s}_0)$, the kriging variance is

$$\sigma_k^2(\vec{s}_0) = E(Z(\vec{s}_0) - \hat{p}(\vec{Z}, \vec{s}_0))^2 = C(\vec{0}) + \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j C(\vec{s}_i - \vec{s}_j) - 2\sum_{i=1}^{n} \lambda_i C(\vec{s}_0 - \vec{s}_i),$$
$$C(\vec{0}) + \vec{\lambda}^T \vec{C}\vec{\lambda} - 2\vec{\lambda}^T \vec{c} = C(\vec{0}) + \vec{\lambda}^T (\vec{c} + \vec{m}) - 2\vec{\lambda}^T \vec{c} = C(\vec{0}) + \vec{\lambda}^T \vec{m} - \vec{\lambda}^T \vec{c} = C(\vec{0}) - \vec{\lambda}^T \vec{c} + m.$$

So,

$$\sigma_k^2(\vec{s}_0) = C(\vec{0}) - \vec{\lambda}^T \vec{c} + m. \tag{2.8}$$

Trans-Gaussian Kriging

The ordinary kriging predictor is unbiased and so $E(Y(\vec{s}_0)) = E(\hat{Y}(\vec{s}_0)) = \mu_Y$. Define $\breve{Z}(\vec{s}_0) = \breve{p}_Z(\vec{Z},\vec{s}_0) = \phi(\hat{p}_Y(\vec{Y},\vec{s}_0)) - B$, where $\hat{p}_Y(\vec{Y},\vec{s}_0)$ is a regular kriging predictor on the $\vec{Y}$ and B is a constant, such that the predictor is unbiased: $E(\breve{Z}(\vec{s}_0) = E(Z(\vec{s}_0))$.

We must define B so that the condition of unbiasedness is fulfilled.

We shall now find B such that the predictor is unbiased. Expanding ϕ around μ_Y we get

$$Z = \phi(Y) \approx \phi(\mu_Y) + (Y-\mu_Y)\phi''(\mu_Y) + \frac{(Y-\mu_Y)^2}{2}\phi''(\mu_y).$$

Therefore,

$$E(Z) \approx \phi(\mu_Y) + \phi''(\mu_y)\frac{E((Y-\mu_Y)^2)}{2} = \phi(\mu_Y) + \frac{\phi''(\mu_y)}{2}Var(Y). \tag{2.9}$$

Similarly, if we denote $Z(s_0)$ by Z_0

$$\begin{aligned} E(\breve{Z}_0) &= E(\phi(\hat{Y}_0)) - B \approx E\left((\phi(\mu_Y) + (\hat{Y}_0 - \mu_Y)\phi''(\mu_Y) + \frac{(\hat{Y}_0-\mu_Y)^2}{2}\phi''(\mu_y) \right) - B = \\ &= \phi(\mu_Y) + \left(E(\hat{Y}_0 - \mu_Y\right)\phi''(\mu_Y) + \frac{\phi''(\mu_y)}{2}E\left((\hat{Y}_0-\mu_Y)^2\right) - B = \\ &= \phi(\mu_Y) + \frac{\phi''(\mu_y)}{2}E\left((\hat{Y}_0-\mu_Y)^2\right) - B. \end{aligned} \tag{2.10}$$

From (2.9) and (2.10), the unbiasedness condition becomes

$$\phi(\mu_Y) + \frac{\phi''(\mu_y)}{2}Var(Y) = \phi(\mu_Y) + \frac{\phi''(\mu_y)}{2}E\left((\hat{Y}_0-\mu_Y)^2\right) - B.$$

So the bias B is

$$B = \frac{\phi''(\mu_y)}{2}E\left((\hat{Y}_0-\mu_Y)^2\right) - \frac{\phi''(\mu_y)}{2}Var(Y). \tag{2.11}$$

It remains to compute $E\left((\hat{Y}_0-\mu_Y)^2\right)$. By definition (2.2),

$$\hat{Y}_0 = \vec{\lambda}_Y^T Y.$$

Therefore,

$$\begin{aligned} E\left((\hat{Y}_0-\mu_Y)^2\right) &= E\left((\vec{\lambda}_Y^T Y - \mu_Y)^2\right) = E\left((\vec{\lambda}_Y^T Y)^2 - 2\vec{\lambda}_Y^T Y\mu_Y + {\mu_Y}^2\right) = \\ &= E\left(\left(\sum_{i=1}^n \lambda_i Y_i\right)^2\right) - 2\mu_Y E\left(\sum \lambda_i Y_i\right) + \mu_Y^2 = \sum_i\sum_j \lambda_i\lambda_j E(Y_iY_j) - \mu_Y^2 = \\ &= \sum_i\sum_j \lambda_i\lambda_j E(\vec{C}_{ij} + \mu_Y^2) - \mu_Y^2 = \sum_i\sum_j \lambda_i\lambda_j\vec{C}_{ij} = \vec{\lambda}_Y^T C_Y \vec{\lambda}_T. \end{aligned}$$

From (2.11):

$$B = \phi''(\mu_y)\left(\frac{\vec{\lambda}_Y^T C_Y \vec{\lambda}_T}{2} - \frac{Var(Y)}{2} \right).$$

Since $\vec{C}\vec{\lambda} = \vec{c} + \vec{m}$ (see (2.5)),

$$B = \frac{\phi''(\mu_y)}{2}\left(\vec{\lambda}_Y^T (\vec{c}_Y + \vec{m}_Y) - Var(Y) \right) = \frac{\phi''(\mu_y)}{2}\left(\vec{\lambda}_Y^T \vec{c}_Y + m_Y) - Var(Y) \right) =$$
$$= \frac{\phi''(\mu_y)}{2}\left(\vec{\lambda}_Y^T \vec{c}_Y + m_Y) - C(\vec{0}) \right).$$

Recall that from (2.8),

$$B = -\phi''(\mu_y)\left(\frac{\sigma_{k,Y}^2(\vec{s}_0)}{2} - m_Y) \right).$$

Therefore, the approximately unbiased predictor is

$$\breve{p}_{\vec{Z}}(\vec{Z}, \vec{s}_0) = \phi(\hat{p}_Y(\vec{Y}, \vec{s}_0)) + \phi''(\mu_y)\left(\frac{\sigma_Y^2(\vec{s}_0)}{2} - m_Y \right).$$

Note, that we do not know μ_y, therefore we use its MLE instead. The likelihood of μ_y is given by

$$\log L(\mu / Y) = \log\left(\frac{1}{\sqrt{2\pi}\left|\vec{C}\right|} \right) - \frac{1}{2}(Y - \mu)^T \vec{C}^{-1}(\vec{Y} - \mu) =$$
$$= \log\left(\frac{1}{\sqrt{2\pi}\left|\vec{C}\right|} \right) - \frac{1}{2}\sum_i \sum_j (\mu^2 - \mu(Y_i + Y_j) + Y_i Y_j)(\vec{C}^{-1})_{ij}.$$

Differentiating, we get

$$\frac{\partial \log L(\mu / Y)}{\partial \mu} = -\frac{1}{2}\sum_i \sum_j (2\mu - Y_i + Y_j)(\vec{C}^{-1})_{ij}.$$

Equating to zero,

$$\mu = \frac{1}{2}\frac{\sum_i \sum_j (Y_i + Y_j)(\vec{C}^{-1})_{ij}}{\sum_i \sum_j (\vec{C}^{-1})_{ij}} = \frac{\sum_i (\vec{C}^{-1}\vec{Y})_i}{\sum_i \sum_j (\vec{C}^{-1})_{ij}}. \tag{2.12}$$

So, the final form of the trans-Gaussian predictor is

$$\breve{p}_{\vec{Z}}(\vec{Z}, \vec{s}_0) = \phi(\hat{p}_Y(\vec{Y}, \vec{s}_0)) + \phi''(\mu_y)\left(\frac{\sigma_Y^2(\vec{s}_0)}{2} - m_Y \right),$$

where

$$\hat{\mu}_Y = \frac{\sum_i (\vec{C}_Y^{-1}\vec{Y})_i}{\sum_i \sum_j (\vec{C}_Y^{-1})_{ij}},\ m_Y = \frac{1-\sum_i (\vec{C}_Y^{-1}\vec{c}_Y)_i}{\sum_i \sum_j (\vec{C}_Y^{-1})_{ij}},\ \sigma_Y^2(\vec{s}_0) = C(\vec{0}) - \vec{\lambda}_Y^{\ T}\vec{c}_Y + m_Y .$$

By definition, the mean squared prediction error of $\breve{Z}(\vec{s}_0)$ is

$$\sigma_{k,Z}^2(\vec{s}_0) = E(\breve{Z}(\vec{s}_0) - Z(\vec{s}_0))^2 = E(\phi(\hat{Y}(\vec{s}_0)) - B - \phi(Y(\vec{s}_0)))^2 \approx$$
$$\approx E\left(\hat{Y}(\vec{s}_0)\phi''(\mu_Y) - Y(\vec{s}_0) - B)\right)^2 = (\phi''(\mu_Y))^2\sigma_{k,Y}^2(\vec{s}_0) + B^2.$$

Attachment 3

Generated Data

The Circulant Embedding method is applicable for generating realizations of stationary Gaussian fields over regular grids (the grid is regular if it has constant x and y steps). Under these conditions, the covariance matrix of the resulting vector $\vec{Z} \sim N(\vec{0}_{n1n2}, \vec{C})$ is block Toeplitz with Toeplitz blocks. By definition, a regular Toeplitz matrix has constant values along all the diagonals.

A block Toeplitz matrix with Toeplitz blocks is defined as a block matrix in which the Toeplitz structure is applicable to the whole blocks and also within each block.

Related to Toeplitz matrices is the class of circulant matrices. A circulant

matrix is a particular case of the Toeplitz matrix, in which all the columns starting with the second one are obtained by circulation of the previous column (a circulation is a cyclical permutation of the vector which moves its last element to the first position and shifts all the others down by one).

Similarly to the Toeplitz case, a block circulant matrix with circulant blocks is defined as a block matrix in which the circulant structure applies to the whole blocks and also within each block. The idea of the Circulant Embedding method is to embed a block Toeplitz matrix into a larger block circulant matrix with circulant blocks, using the standard embedding procedure, that we illustrate in the following example. Suppose we have the symmetric block Toeplitz matrix

$$\vec{C} = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 2 & 7 & 4 & 5 \\ 3 & 2 & 1 & 8 & 7 & 4 \\ \hline 4 & 7 & 8 & 1 & 2 & 3 \\ 5 & 4 & 7 & 2 & 1 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{array}\right) = \begin{array}{cc} \vec{C}^{(1)} & \vec{C}^{(2)} \\ \vec{C}^{(2)^T} & \vec{C}^{(1)} \end{array}$$

First, we embed each of the two blocks $\vec{C}^{(1)}$ and $\vec{C}^{(2)}$ into $\vec{V}^{(1)}$ and $\vec{V}^{(2)}$ respectively. The first column of $\vec{V}^{(1)}$ is obtained as follows: write out all the entries of the first column of $\vec{C}^{(1)}$, a zero, and all the entries of the first row of $\vec{C}^{(1)}$ backwards (without repeating C_{11}). The rest of the $\vec{V}^{(1)}$ is then defined by circulation:

$$\vec{V}^{(1)} = \begin{pmatrix} 1 & 2 & 3 & 0 & 3 & 2 \\ 2 & 1 & 2 & 3 & 0 & 3 \\ 3 & 2 & 1 & 2 & 3 & 0 \\ 0 & 3 & 2 & 1 & 2 & 3 \\ 3 & 0 & 3 & 2 & 1 & 2 \\ 2 & 3 & 0 & 3 & 2 & 1 \end{pmatrix} \quad \vec{V}^{(2)} = \begin{pmatrix} 4 & 5 & 0 & 6 & 8 & 7 \\ 7 & 4 & 5 & 0 & 6 & 8 \\ 8 & 7 & 4 & 5 & 0 & 6 \\ 0 & 8 & 7 & 4 & 5 & 0 \\ 6 & 0 & 8 & 7 & 4 & 5 \\ 5 & 6 & 0 & 8 & 7 & 4 \end{pmatrix}$$

Finally, we combine the blocks $\vec{V}^{(i)}$ into the matrix $\vec{V}$ like that (note that everything is actually defined by the first block column):

$$\vec{V} = \begin{matrix} \vec{V}^{(1)^T} & \vec{V}^{(2)} & \vec{0}_{6x6} & \vec{V}^{(2)^T} \\ \vec{V}^{(2)^T} & \vec{V}^{(1)^T} & \vec{V}^{(2)} & \vec{0}_{6x6} \\ \vec{0}_{6x6} & \vec{V}^{(2)^T} & \vec{V}^{(1)^T} & \vec{V}^{(2)} \\ \vec{V}^{(2)} & \vec{0}_{6x6} & \vec{V}^{(2)^T} & \vec{V}^{(1)^T} \end{matrix}$$

The main advantage of circulant matrices is the following Diagonalization Theorem: if $\vec{V}$ is block circulant with circulant blocks, then

$$\vec{V} = \vec{F}^H \vec{\Lambda} \vec{F}, \tag{3.1}$$

where $\vec{\Lambda}$ is the diagonal matrix of the eigenvalues of $\vec{V}$, and $\vec{F}$ is the matrix of the two-dimensional Fourier transform defined by

$$F_{lm} = \frac{1}{\sqrt{n_1 n_2}} \exp\left(-\frac{2\pi i}{n_1} \left[\frac{l}{n_2} \right] \left[\frac{m}{n_2} \right] \right) \exp\left(-\frac{2\pi i}{n_2} (l \bmod n_2)(m \bmod n_2) \right), l, m = 0, 1, \ldots, n_1 n_2 - 1$$

From (3.1) it follows that $\vec{\lambda}$, the vector of the eigenvalues of $\vec{V}$, is given by $\vec{\lambda} = \sqrt{n_1 n_2} \vec{F} \vec{V}_1$, where $\vec{V}_1$ is the first column of $\vec{V}_1$. Now for a non-negative definite symmetric block circulant matrix $\vec{V}$ with circulant blocks we can generate two independent realizations $\vec{W}^{(1)}, \vec{W}^{(2)} \sim N(\vec{0}, \vec{V})$ by setting

$$\delta_i, \varepsilon_i \sim N(0, \lambda_i), i = 1, \ldots, n_1 n_2, \; \vec{W}^{(1)} = \mathrm{Re}(\vec{F}(\vec{\varepsilon} + i\vec{\delta}), \; \vec{W}^{(2)} = \mathrm{Im}(\vec{F}(\vec{\varepsilon} + i\vec{\delta}).$$

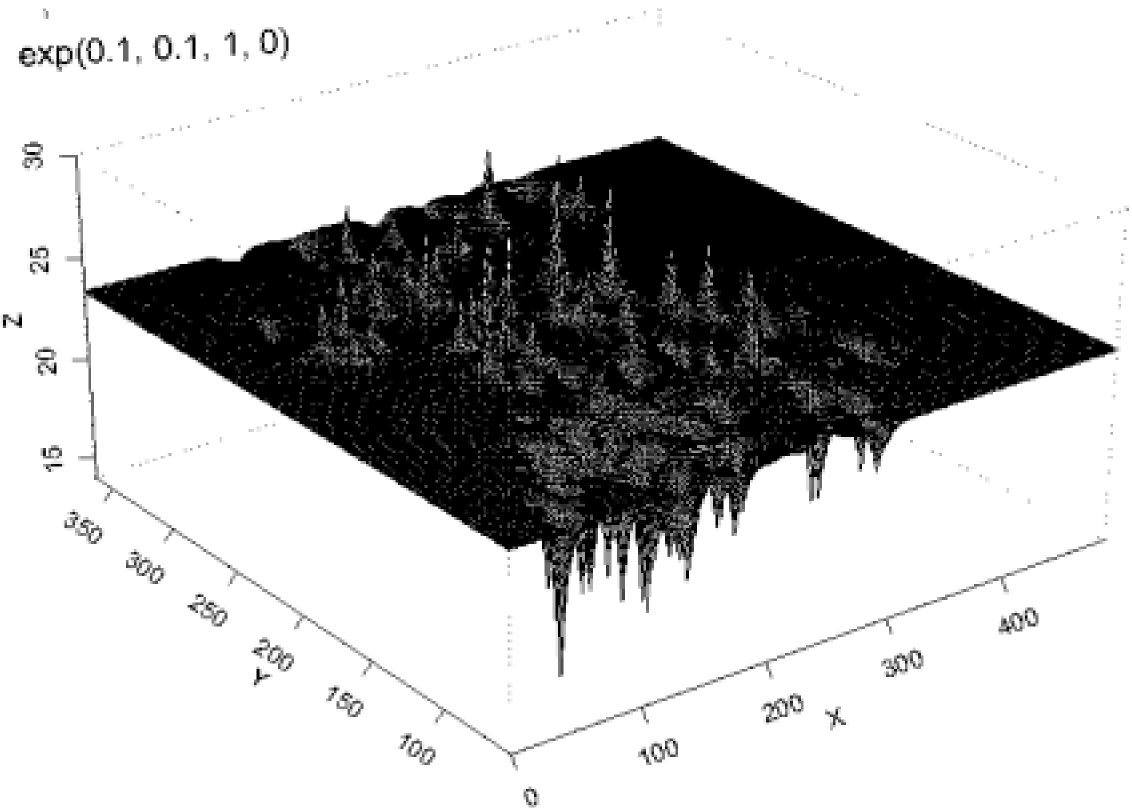
Finally, we use the components of $\vec{W}^{(1)}$ and $\vec{W}^{(2)}$ that correspond to the original field to create the vectors $\vec{Z}^{(1)}$ and $\vec{Z}^{(2)}$ using the following procedure: 1) take the first $2n_1 n_2$ elements of $\vec{W}$; 2) divide it into $2n_1$ groups of length n_2 each; 3) use every other group starting with the first one to populate an $n_1 x n_2$ matrix row-wise; this matrix is the desired sample.

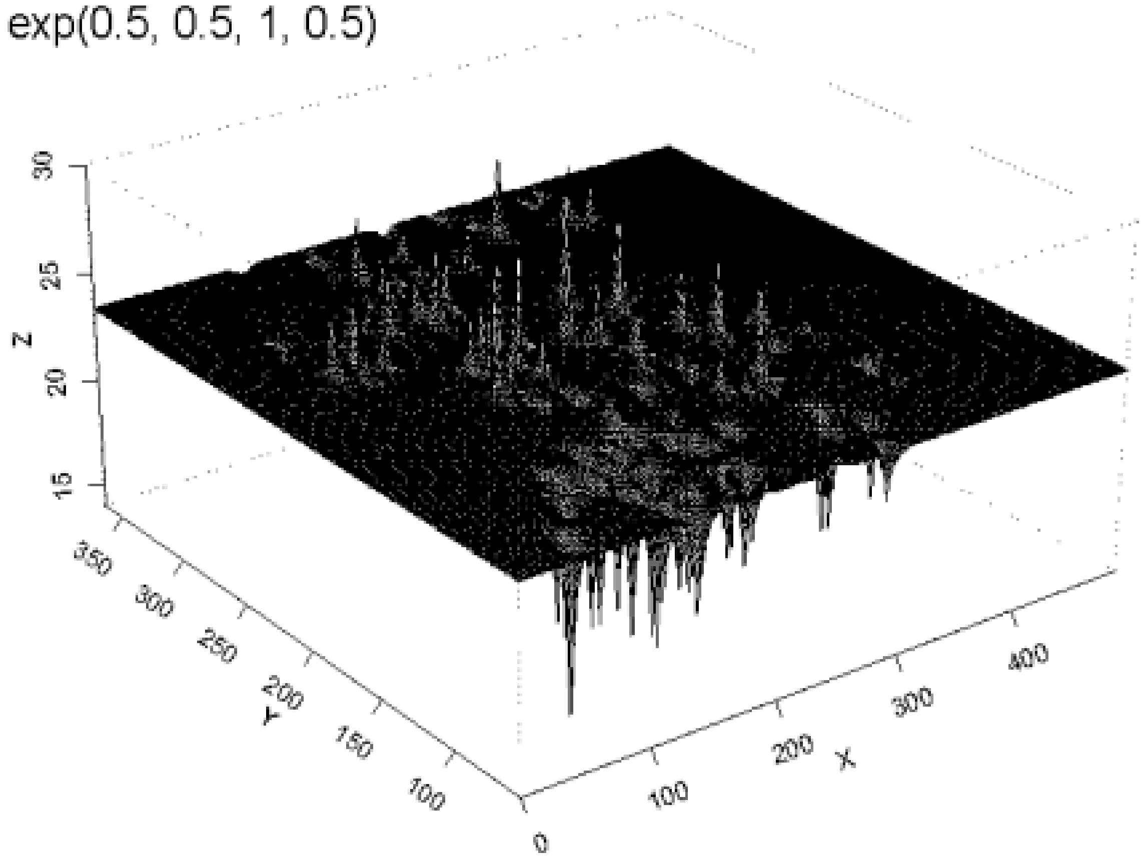
Therefore the Circulant Embedding algorithm is as follows. Given an $n_1 x n_2$

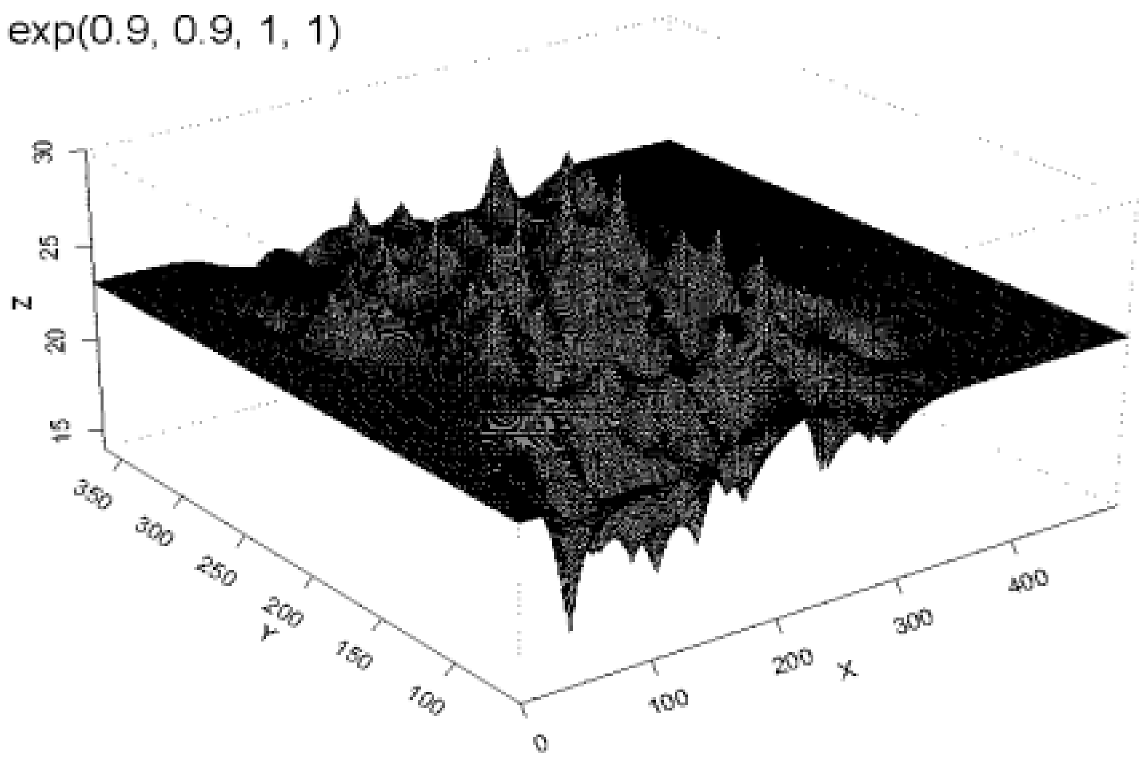
regular grid and a covariance function,

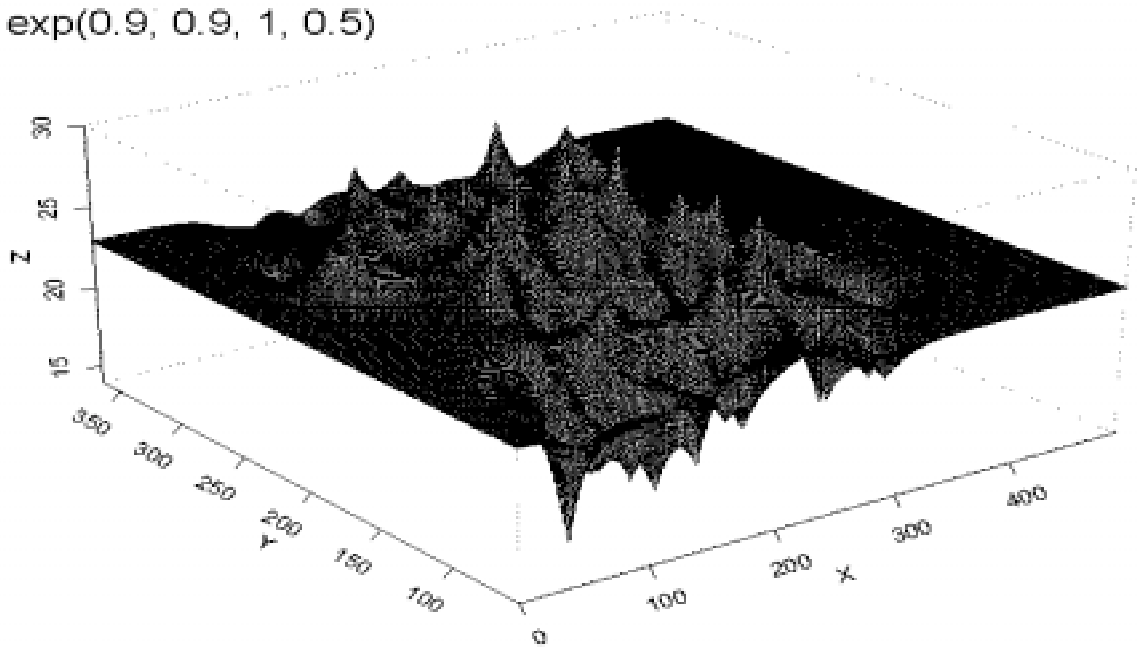
- construct $n_1 n_2 x n_1 n_2$ block Toeplitz covariance matrix $\vec{C}$;
- embed $\vec{C}$ into $4n_1 n_2 x 4n_1 n_2$ block circulant matrix $\vec{V}$;
- generate $\vec{W}^{(1)}, \vec{W}^{(2)} \sim N(\vec{0}_{n1n2}, \vec{V})$;
- extract $\vec{Z}^{(1)}, \vec{Z}^{(2)} \sim N(\vec{0}_{n1n2}, \vec{C})$ from $\vec{W}^{(1)}$ and $\vec{W}^{(2)}$.

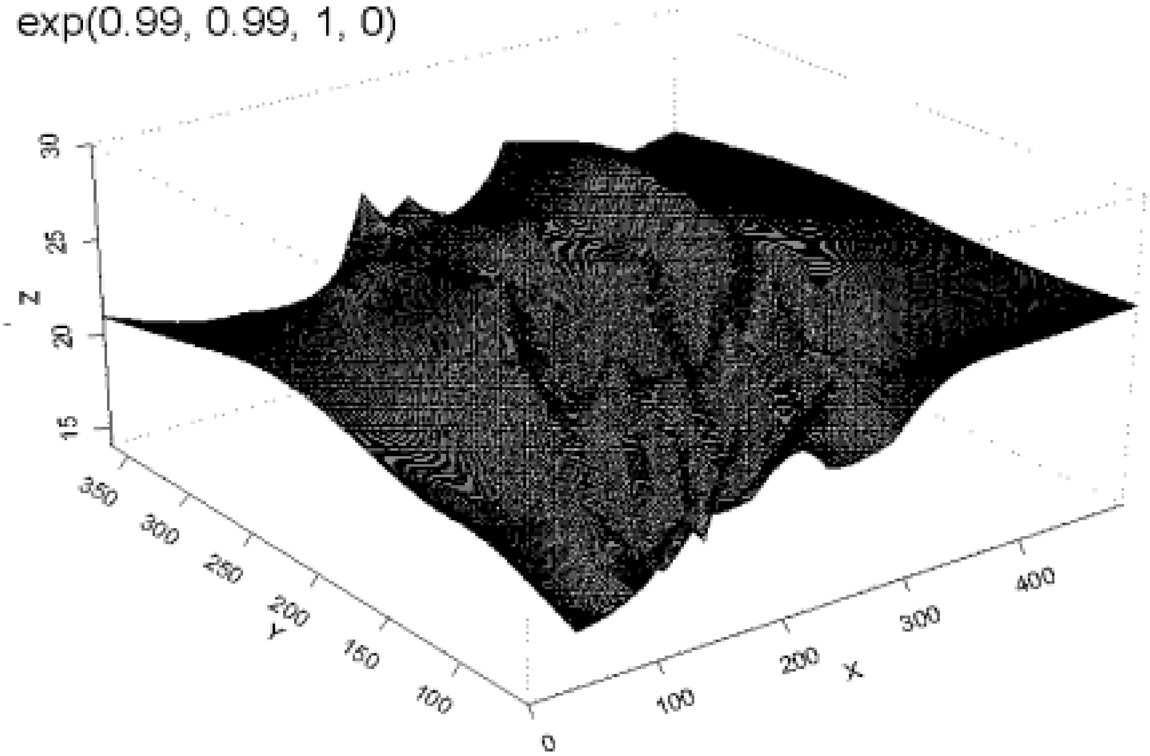
Attachment 4

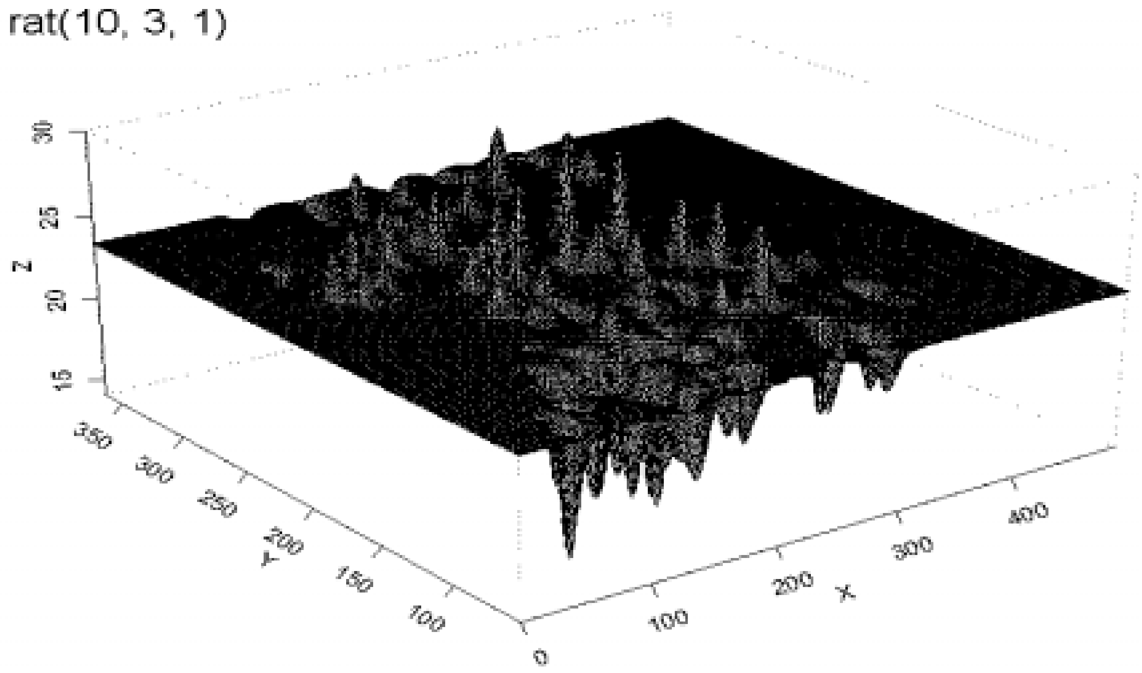


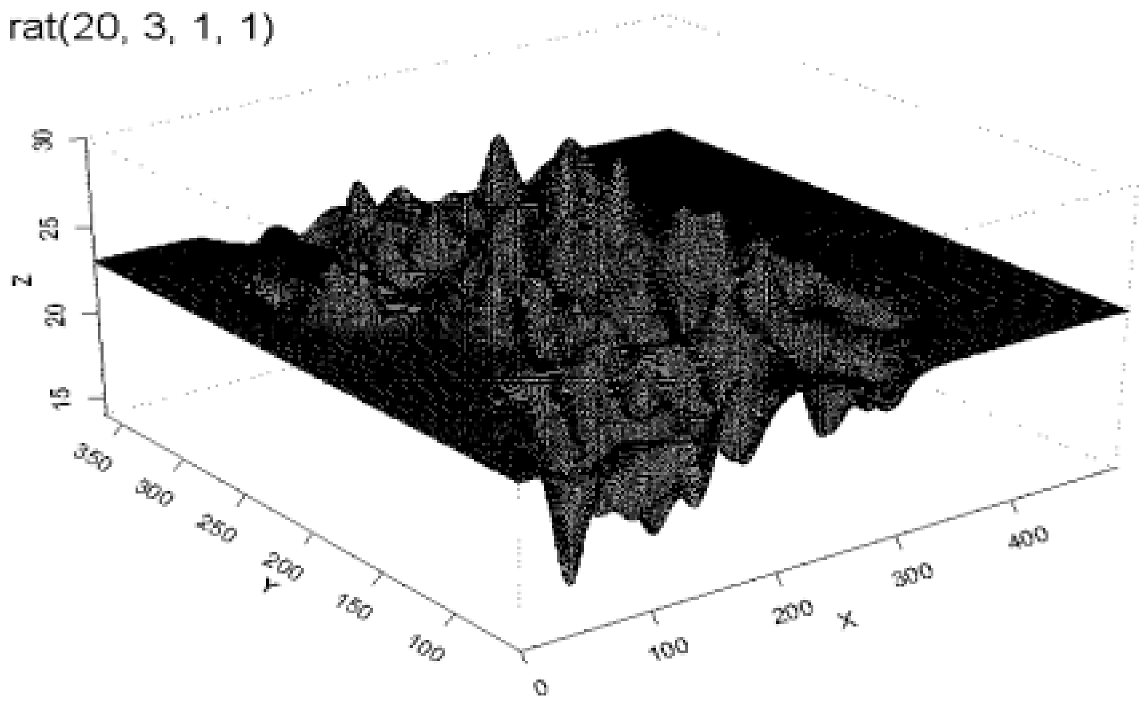

exp(0.5, 0.5, 1, 0.5)
Z
15
20
25
30
Y
350
300
250
200
150
100
X
0
100
200
300
400

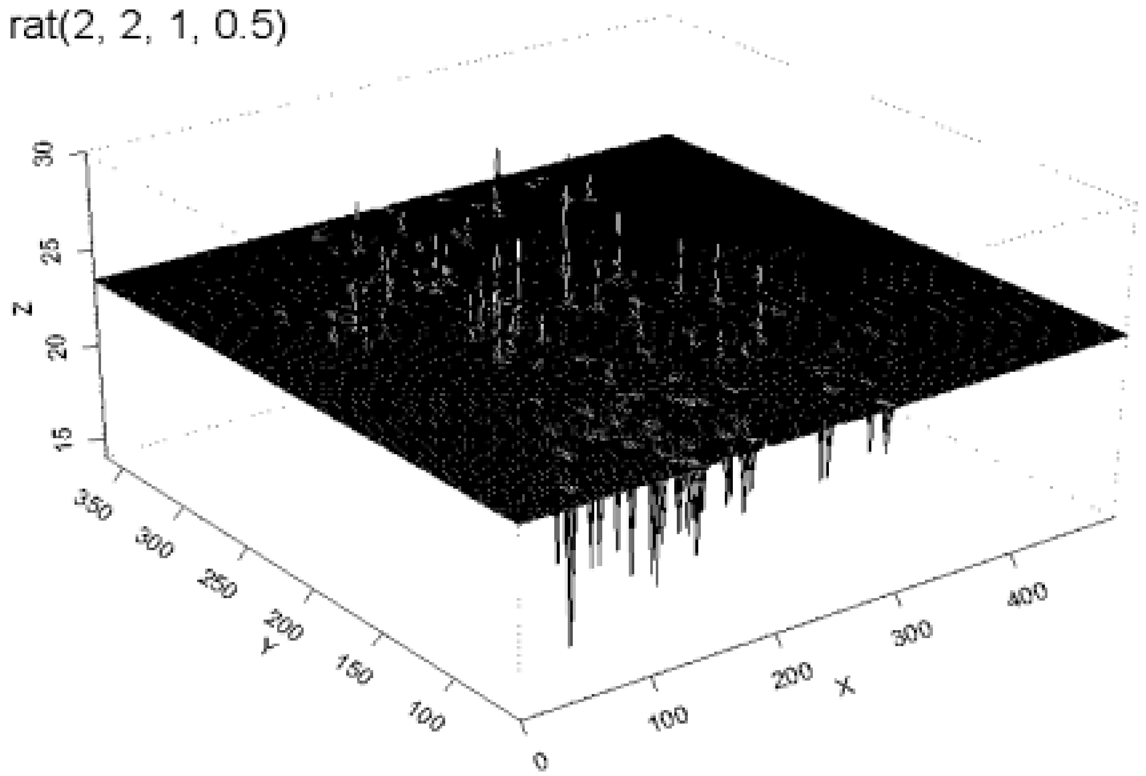

exp(0.9, 0.9, 1, 1)
Z
15
20
25
30
Y
350
300
250
200
150
100
X
0
100
200
300
400

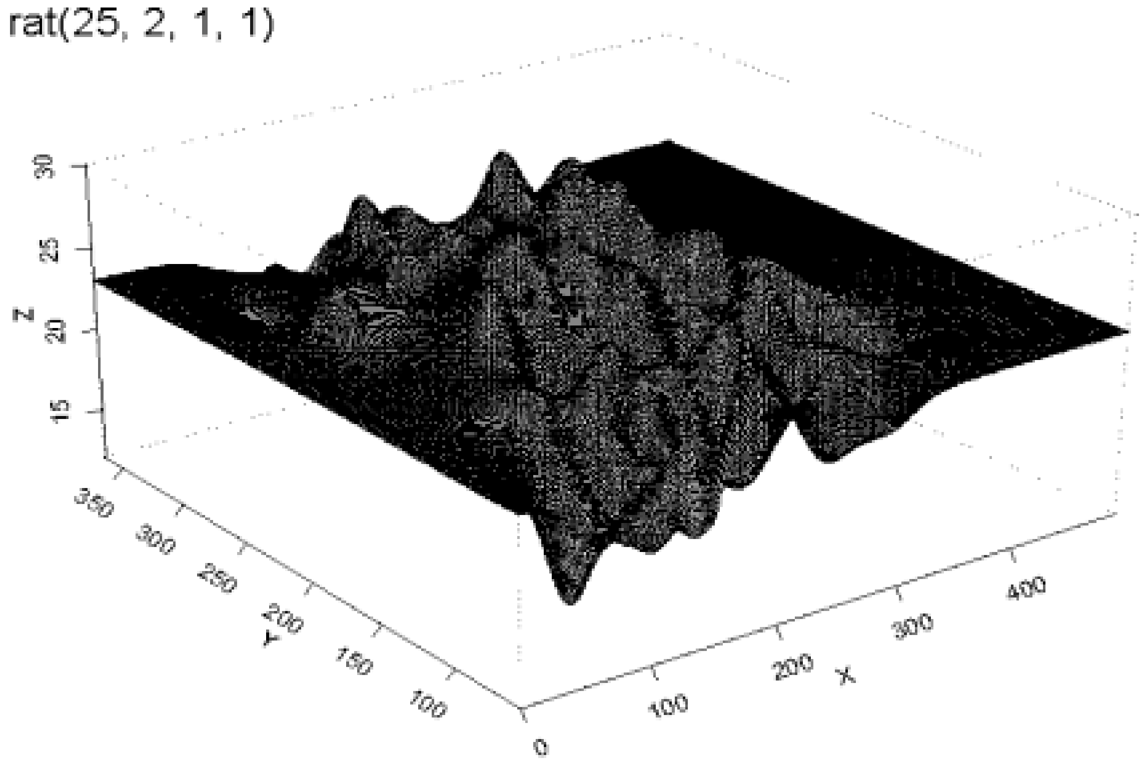

exp(0.9, 0.9, 1, 0.5)
Z
X
Y

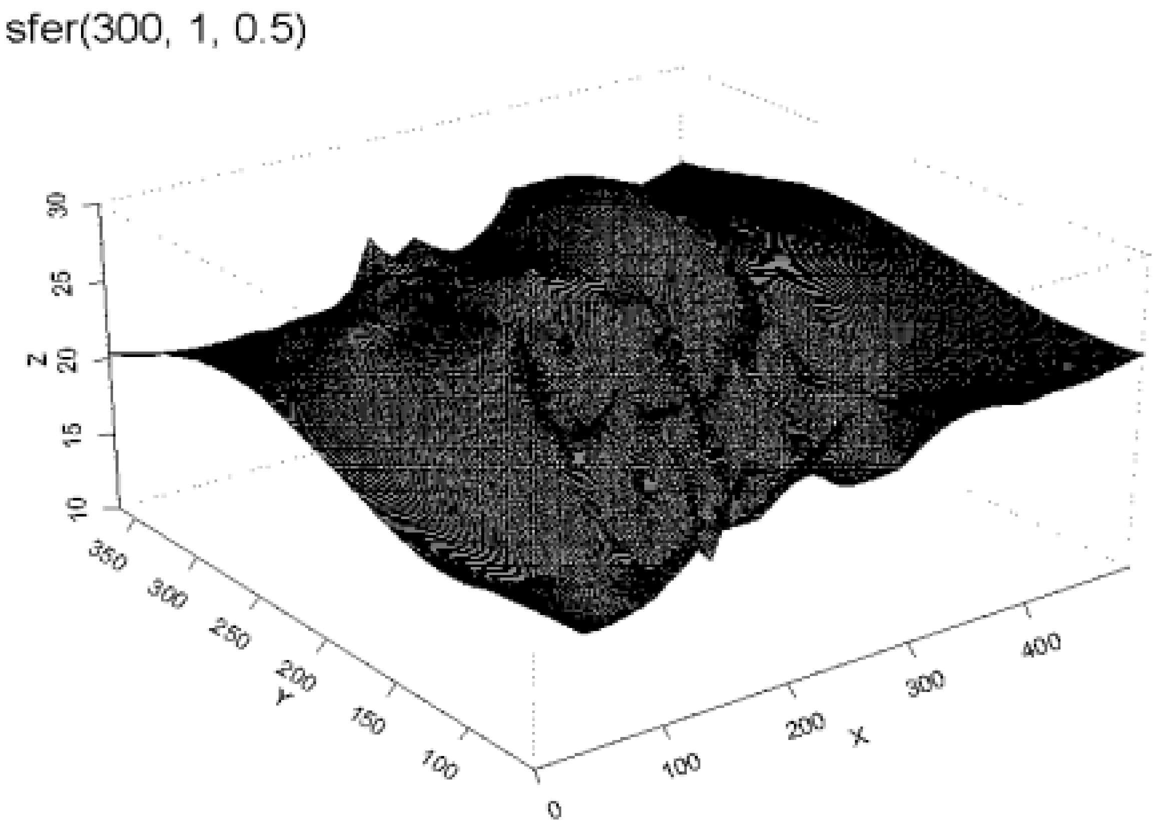

exp(0.99, 0.99, 1, 0)
Z
X
Y

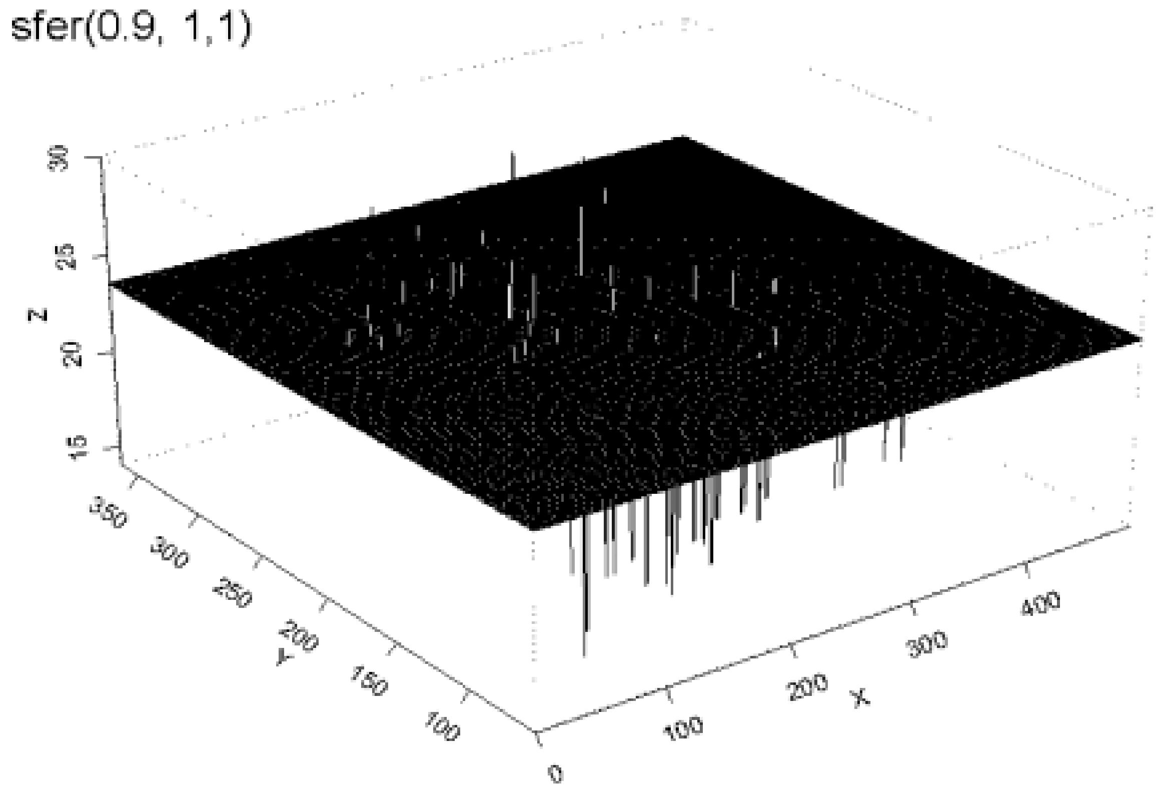

rat(10, 3, 1)
Z
15
20
25
30
Y
350
300
250
200
150
100
0
X
100
200
300
400

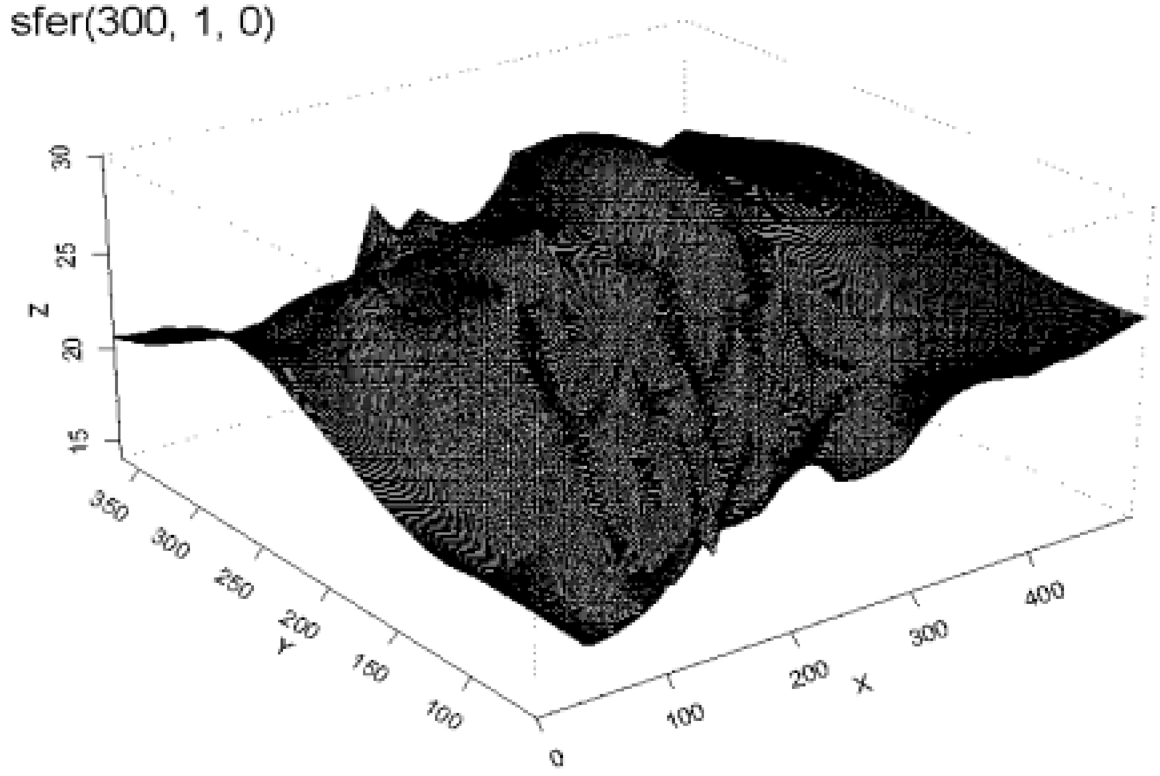

rat(20, 3, 1, 1)
Z
15
20
25
30
Y
350
300
250
200
150
100
0
X
100
200
300
400

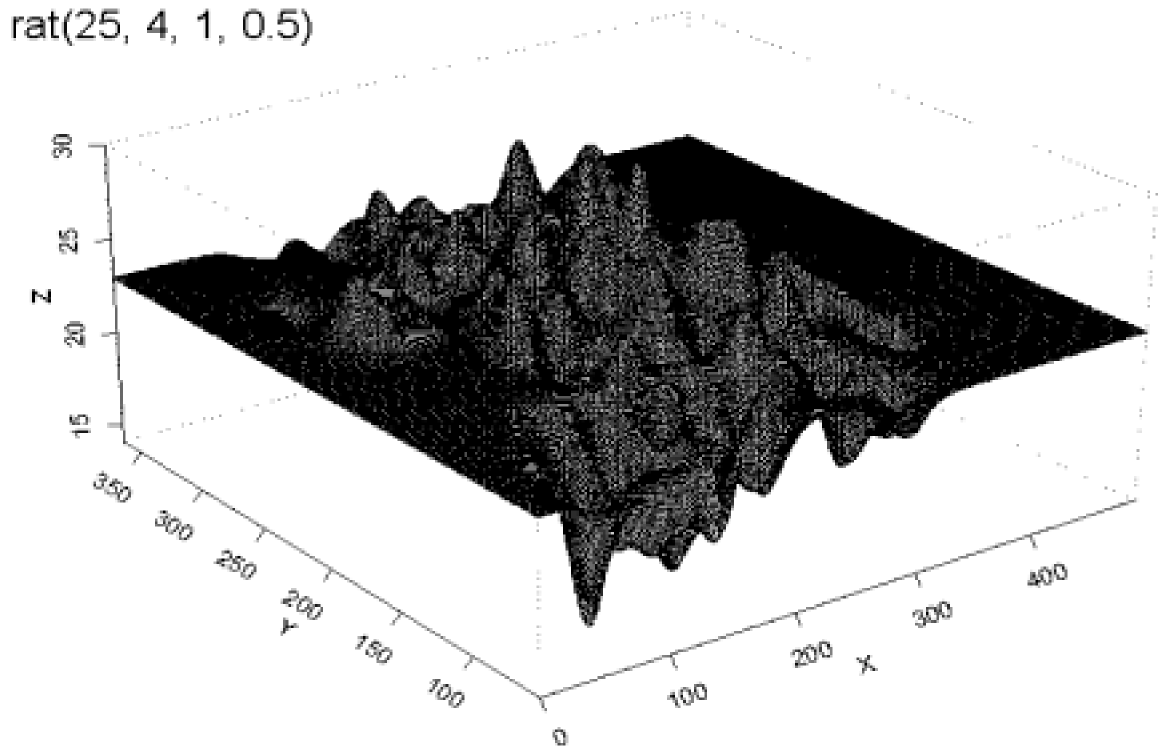

rat(2, 2, 1, 0.5)
Z
15
20
25
30
Y
350
300
250
200
150
100
0
100
200
300
400
X

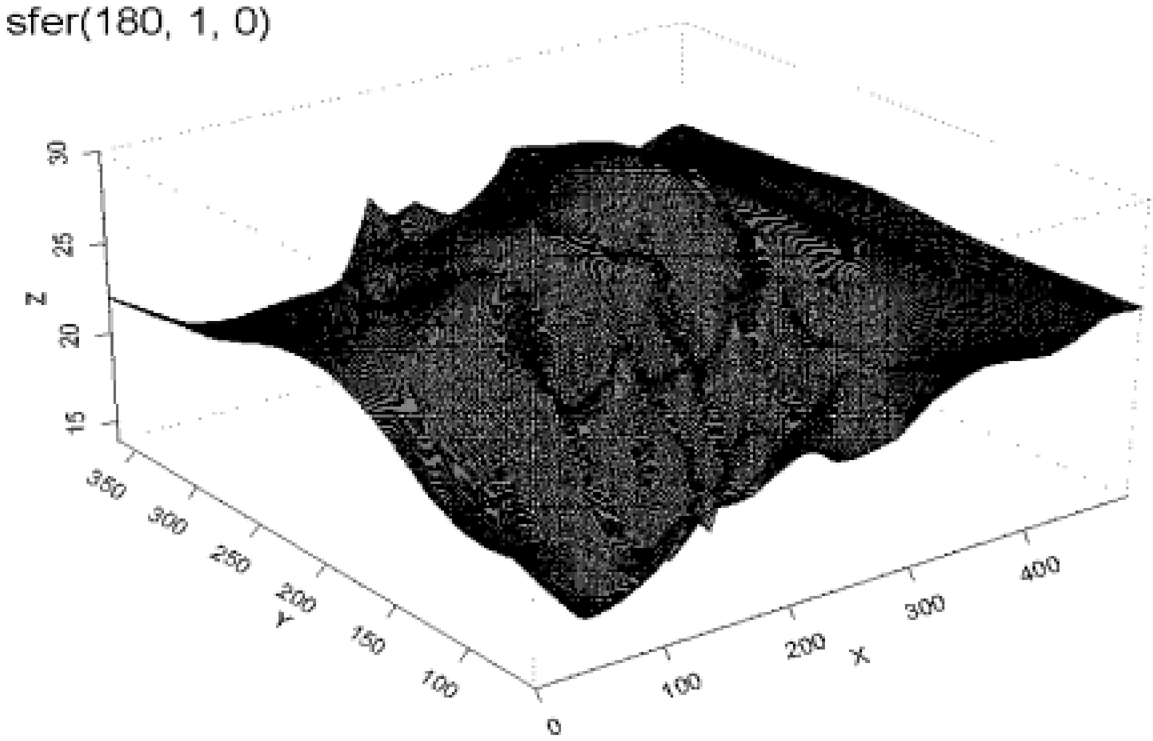

rat(25, 2, 1, 1)
Z
15
20
25
30
Y
350
300
250
200
150
100
0
100
200
300
400
X

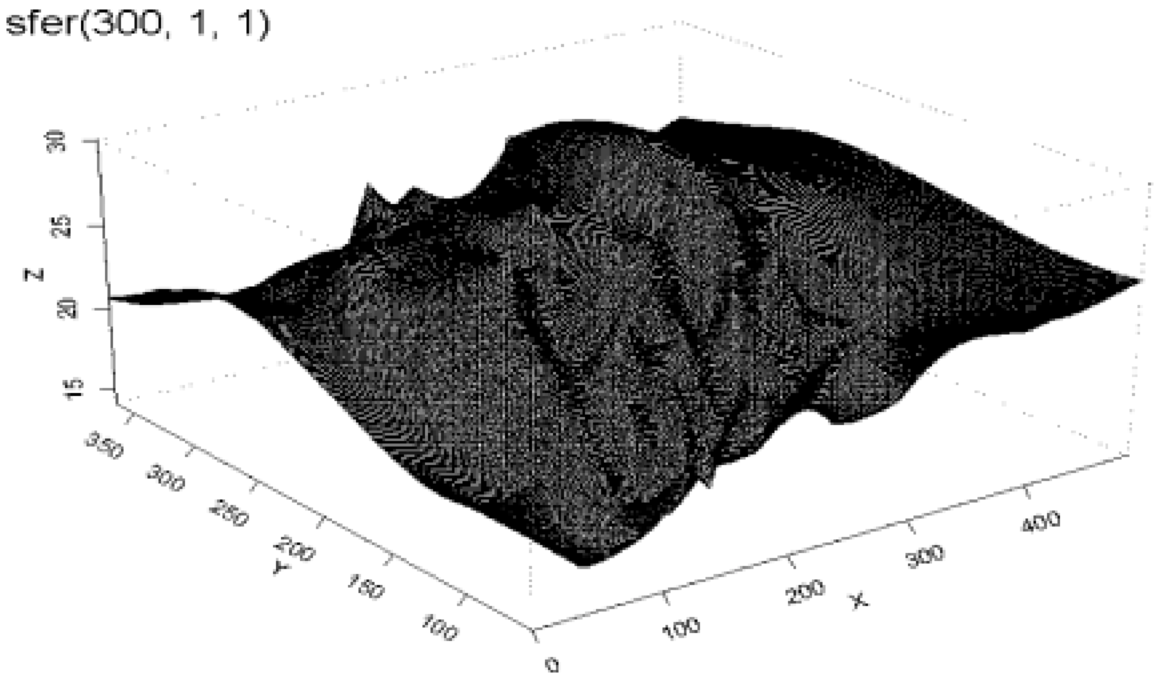

sfer(300, 1, 0.5)
Z
10
15
20
25
30
Y
100
150
200
250
300
350
X
0
100
200
300
400

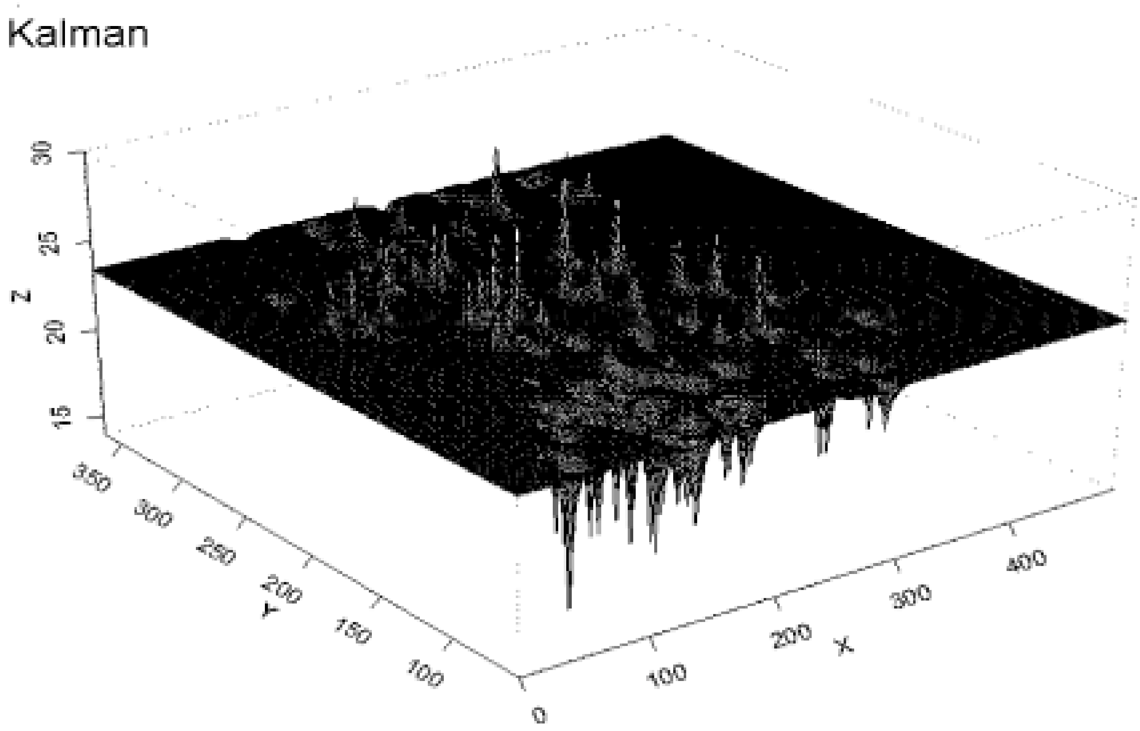

sfer(0.9, 1,1)
Z
15
20
25
30
Y
100
150
200
250
300
350
X
0
100
200
300
400


sfer(300, 1, 0)
Z
15
20
25
30
Y
350
300
250
200
150
100
X
0
100
200
300
400


rat(25, 4, 1, 0.5)
Z
15
20
25
30
Y
350
300
250
200
150
100
X
0
100
200
300
400


sfer(180, 1, 0)
Z
15
20
25
30
350
300
250
200
Y
150
100
0
100
200
300
400
X


sfer(300, 1, 1)
Z
15
20
25
30
350
300
250
200
Y
150
100
0
100
200
300
400
X


Kalman
Z
30
25
20
15
Y
350
300
250
200
150
100
X
0
100
200
300
400

Attachment 5

Data	Maturity	Interest	r-10-3-1						r-2-2-1-0.5					
			"prediction"	"leftend"	"rightend"	L	Var	out	"prediction"	"leftend"	"rightend"	L	Var	out
10-Feb-00	217	34	23.6856	21.7191	25.6522	3.9331	103.7003	0	23.6107	14.1056	33.1158	19.0101	105.2324	0
21-Feb-00	140	25	25.0507	23.1917	26.9096	3.7178	0.3032	1	23.6348	14.1304	33.1393	19.0088	0.7485	1
16-Mar-00	133	26	23.3460	21.3810	25.3111	3.9301	6.7125	0	23.6065	14.1015	33.1115	19.0101	5.4307	1
11-Apr-00	189	21	23.1400	21.2132	25.0667	3.8534	3.9333	0	23.6034	14.0986	33.1082	19.0096	5.9862	1
20-Apr-00	56	17	23.4543	21.4869	25.4217	3.9348	39.7312	0	23.6070	14.1019	33.1121	19.0102	41.6803	1
01-Jun-00	294	18	23.4138	21.4467	25.3810	3.9343	34.6267	0	23.6065	14.1014	33.1116	19.0102	36.9317	1
15-Jun-00	140	17	23.2063	21.2402	25.1725	3.9323	34.4705	0	23.6034	14.0984	33.1085	19.0101	39.2910	1
04-Jul-00	133	19	23.4041	21.4369	25.3713	3.9343	19.4704	0	23.6060	14.1009	33.1111	19.0102	21.2928	1
27-Jul-00	294	27	23.4461	21.4797	25.4126	3.9329	11.2499	0	23.6070	14.1020	33.1121	19.0101	10.1965	1
24-Aug-00	336	20	23.4593	21.4920	25.4266	3.9347	9.9567	0	23.6072	14.1021	33.1122	19.0102	10.9116	1
28-Sep-00	315	27	23.4873	21.5201	25.4545	3.9344	9.5170	0	23.6075	14.1024	33.1126	19.0102	8.7899	1
07-Nov-00	182	26	23.9311	22.0034	25.8588	3.8553	2.4741	1	23.6138	14.1089	33.1186	19.0097	3.5733	1
14-Dec-00	105	25	23.5208	21.5541	25.4874	3.9333	3.5215	1	23.6079	14.1029	33.1130	19.0101	3.2020	1
01-Feb-01	63	22	23.3515	21.3956	25.3074	3.9118	2.6194	1	23.6059	14.1009	33.1109	19.0100	3.5073	1
01-Mar-01	182	23	23.5530	21.5927	25.5133	3.9206	0.5071	1	23.6087	14.1037	33.1137	19.0100	0.5895	1
05-Apr-01	350	26	23.4985	21.5324	25.4645	3.9321	6.2106	0	23.6075	14.1025	33.1126	19.0101	5.6788	1
17-May-01	364	21	23.3763	21.4105	25.3422	3.9317	6.1756	0	23.6063	14.1012	33.1113	19.0101	7.3715	1
28-Jun-01	364	20	23.4181	21.4509	25.3853	3.9344	14.9660	0	23.6065	14.1014	33.1116	19.0102	16.4594	1
06-Sep-01	364	19	22.7599	20.8096	24.7102	3.9006	11.9737	0	23.5981	14.0931	33.1030	19.0100	18.4769	1
27-Nov-01	294	17	22.0126	20.0833	23.9418	3.8585	27.9167	0	23.5854	14.0805	33.0902	19.0098	47.0105	1
						3.9075	**8.7509**	**75.00%**				**19.0100**	**9.8090**	**5.00%**

			r-20-3-1-1						r-25-4-1-0.5					
			"prediction"	"leftend"	"rightend"	L	Var	out	"prediction"	"leftend"	"rightend"	L	Var	out
10-Feb-00	217	34	25.6214	23.7131	27.5297	3.8165	68.0225	0	25.5008	16.4626	34.5391	18.0766	70.0258	1
21-Feb-00	140	25	27.1087	25.9504	28.2671	2.3167	6.8054	0	27.1494	22.1858	32.1130	9.9272	7.0195	1
16-Mar-00	133	26	21.7404	19.9045	23.5764	3.6720	17.6100	0	21.4092	12.8175	30.0008	17.1833	20.5001	1
11-Apr-00	189	21	22.2908	20.9158	23.6657	2.7499	1.2861	1	22.2866	16.2404	28.3328	12.0924	1.2768	1
20-Apr-00	56	17	23.0362	21.0665	25.0060	3.9395	34.6360	0	22.9471	13.5426	32.3515	18.8089	33.5946	1
01-Jun-00	294	18	22.4688	20.5222	24.4154	3.8932	24.3978	0	22.3692	13.0933	31.6451	18.5518	23.4238	1
15-Jun-00	140	17	20.7782	18.8924	22.6639	3.7714	11.8542	0	20.5766	11.6675	29.4858	17.8183	10.5071	1
04-Jul-00	133	19	22.3674	20.4184	24.3163	3.8979	11.3959	0	22.3380	13.0489	31.6271	18.5782	11.1984	1
27-Jul-00	294	27	23.1075	21.2057	25.0093	3.8036	13.6362	0	23.1245	14.1168	32.1321	18.0153	13.5111	1
24-Aug-00	336	20	23.2256	21.2614	25.1897	3.9283	8.5363	0	23.1619	13.7844	32.5393	18.7550	8.1680	1
28-Sep-00	315	27	23.6558	21.7057	25.6059	3.9002	8.5056	0	23.6359	14.3438	32.9280	18.5842	8.6221	1
07-Nov-00	182	26	25.3028	23.8037	26.8019	2.9981	0.0405	1	25.4710	18.6924	32.2496	13.5572	0.0011	1
14-Dec-00	105	25	23.8987	21.9807	25.8167	3.8360	2.2459	1	23.9035	14.7985	33.0085	18.2100	2.2315	1
01-Feb-01	63	22	22.7354	21.0826	24.3883	3.3057	1.0047	1	22.6498	15.1238	30.1758	15.0519	0.8403	1
01-Mar-01	182	23	23.6870	21.9406	25.4333	3.4927	0.7159	1	23.7087	15.6106	31.8069	16.1962	0.7532	1
05-Apr-01	350	26	23.5341	21.6426	25.4255	3.7829	6.0343	0	23.5720	14.6296	32.5145	17.8849	5.8493	1
17-May-01	364	21	22.5471	20.6598	24.4344	3.7746	2.7418	1	22.4617	13.5450	31.3784	17.8335	2.4664	1
28-Jun-01	364	20	22.6259	20.6737	24.5781	3.9044	9.4641	0	22.5446	13.2418	31.8474	18.6055	8.9705	1
06-Sep-01	364	19	20.6672	18.9824	22.3521	3.3697	1.8705	1	20.4934	12.6998	28.2870	15.5872	1.4252	1
27-Nov-01	294	17	18.4298	16.8667	19.9928	3.1261	2.8928	0	18.0698	10.8975	25.2422	14.3448	1.7980	1
						3.5640	**5.8424**	**65.00%**				**16.6831**	**5.8046**	**0.00%**

			e-0.1-0.1-1						e-0.5-0.5-1-0.5					
			"prediction"	"leftend"	"rightend"	L	Var	out	"prediction"	"leftend"	"rightend"	L	Var	out
10-Feb-00	217	34	23.7752	21.8090	25.7414	3.9324	101.8848	0	23.5515	14.0629	33.0402	18.9773	106.4496	0
21-Feb-00	140	25	24.9154	23.0016	26.8293	3.8278	0.1726	1	23.9876	14.5462	33.4289	18.8827	0.2626	1
16-Mar-00	133	26	23.3537	21.3892	25.3182	3.9291	6.6728	0	23.5007	14.0127	32.9888	18.9761	5.9349	1
11-Apr-00	189	21	23.1501	21.2095	25.0907	3.8812	3.9737	0	23.4549	13.9824	32.9274	18.9449	5.2817	1
20-Apr-00	56	17	23.4902	21.5229	25.4575	3.9346	40.1853	0	23.5234	14.0344	33.0124	18.9780	40.6078	1
01-Jun-00	294	18	23.4380	21.4710	25.4049	3.9340	34.9113	0	23.5190	14.0301	33.0080	18.9779	35.8758	1
15-Jun-00	140	17	23.1888	21.2231	25.1546	3.9315	34.2656	0	23.4869	13.9984	32.9754	18.9770	37.8442	1
04-Jul-00	133	19	23.4322	21.4652	25.3992	3.9340	19.7192	0	23.5197	14.0308	33.0087	18.9779	20.5042	1
27-Jul-00	294	27	23.4784	21.5122	25.4445	3.9322	11.0347	0	23.5226	14.0340	33.0113	18.9773	10.7426	1
24-Aug-00	336	20	23.4944	21.5272	25.4616	3.9344	10.1793	0	23.5236	14.0346	33.0126	18.9780	10.3664	1
28-Sep-00	315	27	23.5288	21.5618	25.4958	3.9340	9.2628	0	23.5272	14.0383	33.0162	18.9779	9.2724	1
07-Nov-00	182	26	23.9081	21.9635	25.8528	3.8893	2.5470	1	23.6430	14.1702	33.1159	18.9457	3.4634	1
14-Dec-00	105	25	23.5646	21.5983	25.5310	3.9327	3.3588	1	23.5332	14.0445	33.0219	18.9774	3.4749	1
01-Feb-01	63	22	23.3842	21.4231	25.3453	3.9222	2.7260	1	23.4978	14.0119	32.9838	18.9719	3.1143	1
01-Mar-01	182	23	23.5922	21.6320	25.5525	3.9205	0.5645	1	23.5490	14.0631	33.0350	18.9719	0.5015	1
05-Apr-01	350	26	23.5349	21.5691	25.5006	3.9315	6.0305	0	23.5330	14.0445	33.0214	18.9769	6.0398	1
17-May-01	364	21	23.4051	21.4395	25.3707	3.9312	6.3195	0	23.5084	14.0200	32.9968	18.9768	6.8498	1
28-Jun-01	364	20	23.4458	21.4788	25.4129	3.9341	15.1814	0	23.5190	14.0301	33.0080	18.9779	15.7571	1
06-Sep-01	364	19	22.9185	20.9625	24.8745	3.9121	13.0966	0	23.3455	13.8631	32.8278	18.9647	16.3694	1
27-Nov-01	294	17	22.4240	20.4774	24.3707	3.8933	32.4342	0	23.1188	13.6452	32.5924	18.9472	40.8306	1
						3.9186	**8.8630**	**75.00%**				**18.9668**	**9.4886**	**5.00%**

			e-0.9-0.9-1-0.5						e-0.9-0.9-1-1					
			"prediction"	"leftend"	"rightend"	L	Var	out	"prediction"	"leftend"	"rightend"	L	Var	out
10-Feb-00	217	34	26.1472	17.0725	35.2218	18.1493	59.6259	1	26.4344	24.5331	28.3357	3.8025	55.2732	0
21-Feb-00	140	25	26.6746	19.1711	34.1781	15.0070	4.7289	1	26.7738	25.2017	28.3458	3.1442	5.1700	0
16-Mar-00	133	26	22.3925	13.5608	31.2241	17.6633	12.5628	1	22.3516	20.5013	24.2020	3.7007	12.8541	0
11-Apr-00	189	21	22.3927	14.5495	30.2360	15.6865	1.5278	1	22.3078	20.6646	23.9511	3.2866	1.3251	1
20-Apr-00	56	17	22.7522	13.3545	32.1499	18.7954	31.3730	1	22.8657	20.8967	24.8346	3.9379	32.6575	0
01-Jun-00	294	18	22.2569	13.0471	31.4667	18.4196	22.3490	1	22.2708	20.3412	24.2003	3.8592	22.4806	0
15-Jun-00	140	17	20.2141	11.2664	29.1618	17.8954	8.2882	1	20.2541	18.3794	22.1287	3.7493	8.5201	0
04-Jul-00	133	19	21.7478	12.5356	30.9600	18.4243	7.5968	1	21.7872	19.8571	23.7173	3.8602	7.8156	0
27-Jul-00	294	27	23.1482	14.1205	32.1760	18.0555	13.3370	1	23.1098	21.2183	25.0012	3.7829	13.6193	0
24-Aug-00	336	20	23.2441	13.9189	32.5693	18.6504	8.6450	1	23.2764	21.3226	25.2301	3.9075	8.8356	0
28-Sep-00	315	27	23.7264	14.4674	32.9855	18.5181	8.0987	1	23.7708	21.8309	25.7107	3.8798	7.8480	0
07-Nov-00	182	26	25.2847	17.1680	33.4013	16.2333	0.0481	1	25.2145	23.5140	26.9151	3.4011	0.0838	1
14-Dec-00	105	25	24.0529	14.9387	33.1671	18.2283	1.8075	1	24.0627	22.1531	25.9722	3.8191	1.7814	1
01-Feb-01	63	22	22.9249	14.1170	31.7328	17.6157	1.4204	1	22.9211	21.0757	24.7665	3.6908	1.4113	1
01-Mar-01	182	23	23.9938	15.4354	32.5521	17.1167	1.3291	1	23.9053	22.1122	25.6984	3.5862	1.1329	1
05-Apr-01	350	26	23.4093	14.3403	32.4783	18.1379	6.6629	1	23.4253	21.5253	25.3254	3.8002	6.5804	0
17-May-01	364	21	22.5785	13.5000	31.6569	18.1569	2.8468	1	22.6026	20.7005	24.5047	3.8041	2.9288	1
28-Jun-01	364	20	22.3217	13.0433	31.6001	18.5568	7.6848	1	22.3813	20.4374	24.3253	3.8879	8.0191	0
06-Sep-01	364	19	20.7891	12.1579	29.4203	17.2624	2.2186	1	20.8432	19.0348	22.6515	3.6167	2.3827	1
27-Nov-01	294	17	19.1867	10.8295	27.5439	16.7145	6.0406	1	19.2791	17.5282	21.0301	3.5019	6.5036	0
						17.6644	**5.2048**	**0.00%**				**3.7009**	**5.1806**	**65.00%**

			e-0.99-0.99-1						s-0.9-0.9-1-1					
			"prediction"	"leftend"	"rightend"	L	Var	out	"prediction"	"leftend"	"rightend"	L	Var	out
10-Feb-00	217	34	32.7071	31.8598	33.5543	1.6945	1.3500	0	23.7230	21.7564	25.6896	3.9332	102.9414	0
21-Feb-00	140	25	26.9182	26.3632	27.4731	1.1099	5.8475	0	23.7230	21.7564	25.6896	3.9332	0.6038	1
16-Mar-00	133	26	20.5459	19.8012	21.2905	1.4893	29.0629	0	23.7230	21.7564	25.6896	3.9332	4.9014	0
11-Apr-00	189	21	22.2400	21.6541	22.8258	1.1716	1.1734	0	23.7230	21.7564	25.6896	3.9332	6.5857	0
20-Apr-00	56	17	19.2107	17.9378	20.4835	2.5457	4.2423	0	23.7230	21.7564	25.6896	3.9332	43.1908	0
01-Jun-00	294	18	20.6041	19.7384	21.4698	1.7314	9.4536	0	23.7230	21.7564	25.6896	3.9332	38.3605	0
15-Jun-00	140	17	17.9043	17.1327	18.6760	1.5434	0.3240	1	23.7230	21.7564	25.6896	3.9332	40.8041	0
04-Jul-00	133	19	19.8115	18.9495	20.6736	1.7241	0.6723	1	23.7230	21.7564	25.6896	3.9332	22.3861	0
27-Jul-00	294	27	23.0654	22.2681	23.8627	1.5945	13.9488	0	23.7230	21.7564	25.6896	3.9332	9.4694	0
24-Aug-00	336	20	23.5680	22.6077	24.5284	1.9207	10.6546	0	23.7230	21.7564	25.6896	3.9332	11.6901	0
28-Sep-00	315	27	25.4344	24.5160	26.3528	1.8368	1.2948	0	23.7230	21.7564	25.6896	3.9332	8.1185	0
07-Nov-00	182	26	26.0484	25.4214	26.6753	1.2539	0.2963	1	23.7230	21.7564	25.6896	3.9332	3.1723	1
14-Dec-00	105	25	26.1331	25.2577	27.0084	1.7507	0.5413	1	23.7230	21.7564	25.6896	3.9332	2.8035	1
01-Feb-01	63	22	23.0219	22.2113	23.8326	1.6214	1.6611	0	23.7230	21.7564	25.6896	3.9332	3.9596	0
01-Mar-01	182	23	24.2850	23.5912	24.9788	1.3875	2.0855	0	23.7230	21.7564	25.6896	3.9332	0.7781	1
05-Apr-01	350	26	24.2992	23.4343	25.1641	1.7298	2.8608	0	23.7230	21.7564	25.6896	3.9332	5.1420	0
17-May-01	364	21	21.3744	20.4611	22.2876	1.8265	0.2334	1	23.7230	21.7564	25.6896	3.9332	8.0188	0
28-Jun-01	364	20	19.8767	18.8963	20.8571	1.9608	0.1071	1	23.7230	21.7564	25.6896	3.9332	17.4177	0
06-Sep-01	364	19	17.9285	17.1544	18.7026	1.5483	1.8798	0	23.7230	21.7564	25.6896	3.9332	19.5664	0
27-Nov-01	294	17	16.3899	15.7045	17.0754	1.3708	0.1149	1	23.7230	21.7564	25.6896	3.9332	48.9165	0
						1.6406	**2.1951**	**65.00%**				**3.9332**	**9.9707**	**80.00%**

			s-300-300-1						s-300-300-1-0.5					
			"prediction"	"leftend"	"rightend"	L	Var	out	"prediction"	"leftend"	"rightend"	L	Var	out
10-Feb-00	217	34	33.0163	32.4113	33.6214	1.2102	0.7270	0	32.7212	29.6899	35.7526	6.0627	1.3173	1
21-Feb-00	140	25	26.8987	26.5048	27.2926	0.7878	5.7538	0	26.7523	24.7790	28.7256	3.9467	5.0729	0
16-Mar-00	133	26	20.4267	19.8968	20.9565	1.0596	30.3625	0	20.6135	17.9592	23.2677	5.3085	28.3385	0
11-Apr-00	189	21	22.2507	21.8348	22.6665	0.8317	1.1967	0	22.2356	20.1523	24.3190	4.1668	1.1641	1
20-Apr-00	56	17	18.1394	17.1826	19.0963	1.9137	0.9769	0	17.9300	13.1364	22.7237	9.5873	0.6069	1
01-Jun-00	294	18	20.6077	19.9903	21.2252	1.2348	9.4762	0	20.6204	17.5273	23.7136	6.1863	9.5546	1
15-Jun-00	140	17	17.9102	17.3607	18.4596	1.0988	0.3306	0	17.8754	15.1229	20.6278	5.5050	0.2918	1
04-Jul-00	133	19	19.8250	19.2098	20.4401	1.2303	0.6946	0	19.7809	16.6992	22.8626	6.1634	0.6230	1
27-Jul-00	294	27	23.0378	22.4700	23.6057	1.1357	14.1555	0	23.0691	20.2242	25.9140	5.6898	13.9214	0
24-Aug-00	336	20	23.4178	22.7293	24.1063	1.3769	9.6963	0	23.4791	20.0300	26.9282	6.8982	10.0818	1
28-Sep-00	315	27	25.3982	24.7416	26.0548	1.3131	1.3785	0	25.4154	22.1261	28.7047	6.5786	1.3384	1
07-Nov-00	182	26	26.0459	25.6006	26.4913	0.8907	0.2936	0	26.0477	23.8166	28.2788	4.4622	0.2955	1
14-Dec-00	105	25	26.0346	25.4087	26.6604	1.2518	0.4060	0	26.0933	22.9577	29.2288	6.2711	0.4843	1
01-Feb-01	63	22	23.0934	22.5111	23.6757	1.1646	1.8503	0	23.0405	20.1232	25.9577	5.8345	1.7092	1
01-Mar-01	182	23	24.2228	23.7296	24.7161	0.9865	1.9097	0	24.2366	21.7655	26.7078	4.9423	1.9481	1
05-Apr-01	350	26	24.4255	23.8068	25.0443	1.2376	2.4493	0	24.4030	21.3031	27.5030	6.1999	2.5203	1
17-May-01	364	21	21.3022	20.6443	21.9601	1.3158	0.1689	1	21.3039	18.0081	24.5997	6.5917	0.1703	1
28-Jun-01	364	20	19.6427	18.9377	20.3478	1.4102	0.0087	1	19.6704	16.1381	23.2027	7.0646	0.0146	1
06-Sep-01	364	19	17.8227	17.2698	18.3757	1.1060	2.1810	0	17.8316	15.0613	20.6019	5.5406	2.1549	1
27-Nov-01	294	17	16.3515	15.8640	16.8389	0.9750	0.1425	1	16.3623	13.9201	18.8044	4.8843	0.1344	1
						1.1765	**2.1040**	**85.00%**				**5.8942**	**2.0436**	**15.00%**

			s-300-300-1-1						s-180-180-1					
			"prediction"	"leftend"	"rightend"	L	Var	out	"prediction"	"leftend"	"rightend"	L	Var	out
10-Feb-00	217	34	33.0163	32.4113	33.6214	1.2102	0.7270	0	32.9980	32.2161	33.7799	1.5638	0.7586	0
21-Feb-00	140	25	26.8987	26.5048	27.2926	0.7878	5.7538	0	26.9082	26.3996	27.4168	1.0172	5.7993	0
16-Mar-00	133	26	20.4267	19.8968	20.9565	1.0596	30.3625	0	20.4762	19.7923	21.1602	1.3679	29.8187	0
11-Apr-00	189	21	22.2507	21.8348	22.6665	0.8317	1.1967	0	22.2266	21.6896	22.7637	1.0741	1.1447	0
20-Apr-00	56	17	18.1394	17.1826	19.0963	1.9137	0.9769	0	17.6995	16.4760	18.9231	2.4471	0.3009	1
01-Jun-00	294	18	20.6077	19.9903	21.2252	1.2348	9.4762	0	20.5909	19.7936	21.3883	1.5947	9.3731	0
15-Jun-00	140	17	17.9102	17.3607	18.4596	1.0988	0.3306	0	17.8556	17.1461	18.5652	1.4191	0.2709	1
04-Jul-00	133	19	19.8250	19.2098	20.4401	1.2303	0.6946	0	19.7595	18.9650	20.5540	1.5889	0.5897	1
27-Jul-00	294	27	23.0378	22.4700	23.6057	1.1357	14.1555	0	22.9098	22.1770	23.6426	1.4656	15.1353	0
24-Aug-00	336	20	23.4178	22.7293	24.1063	1.3769	9.6963	0	23.2642	22.3792	24.1493	1.7700	8.7637	0
28-Sep-00	315	27	25.3982	24.7416	26.0548	1.3131	1.3785	0	25.6063	24.7607	26.4519	1.6912	0.9331	0
07-Nov-00	182	26	26.0459	25.6006	26.4913	0.8907	0.2936	0	25.9212	25.3464	26.4960	1.1496	0.1740	1
14-Dec-00	105	25	26.0346	25.4087	26.6604	1.2518	0.4060	0	26.3814	25.5748	27.1880	1.6132	0.9684	0
01-Feb-01	63	22	23.0934	22.5111	23.6757	1.1646	1.8503	0	23.0235	22.2728	23.7742	1.5014	1.6651	0
01-Mar-01	182	23	24.2228	23.7296	24.7161	0.9865	1.9097	0	24.1133	23.4769	24.7496	1.2727	1.6189	0
05-Apr-01	350	26	24.4255	23.8068	25.0443	1.2376	2.4493	0	24.2879	23.4902	25.0855	1.5953	2.8992	0
17-May-01	364	21	21.3022	20.6443	21.9601	1.3158	0.1689	1	21.3962	20.5472	22.2452	1.6980	0.2550	1
28-Jun-01	364	20	19.6427	18.9377	20.3478	1.4102	0.0087	1	19.8534	18.9416	20.7652	1.8235	0.0924	1
06-Sep-01	364	19	17.8227	17.2698	18.3757	1.1060	2.1810	0	17.9096	17.1936	18.6257	1.4321	1.9319	0
27-Nov-01	294	17	16.3515	15.8640	16.8389	0.9750	0.1425	1	16.3732	15.7432	17.0032	1.2600	0.1266	1
						1.1765	**2.1040**	**85.00%**				**1.5173**	**2.0655**	**65.00%**

			r-25-2-1-1						Kalman					
			"prediction"	"leftend"	"rightend"	L	Var	out	"prediction"	"leftend"	"rightend"	L	Var	out
10-Feb-00	217	34	28.5755	26.9616	30.1894	3.2278	28.0212	0	23.664111	21.697316	25.630906	3.9336	104.1393	0
21-Feb-00	140	25	28.0652	27.4736	28.6568	1.1833	12.7106	0	24.208889	22.255999	26.16178	3.9058	0.0847	1
16-Mar-00	133	26	18.9917	17.6247	20.3587	2.7341	48.2353	0	23.57866	21.612073	25.545246	3.9332	5.5612	0
11-Apr-00	189	21	21.5786	20.8072	22.3501	1.5429	0.1780	1	23.490976	21.529251	25.452702	3.9235	5.4488	0
20-Apr-00	56	17	22.2996	20.3393	24.2599	3.9207	26.5080	0	23.615749	21.648832	25.582665	3.9338	41.7930	0
01-Jun-00	294	18	21.2752	19.5346	23.0158	3.4812	14.0310	0	23.608003	21.641116	25.574891	3.9338	36.9496	0
15-Jun-00	140	17	18.6193	17.1201	20.1186	2.9985	1.6491	1	23.55881	21.592067	25.525554	3.9335	38.7338	0
04-Jul-00	133	19	21.0003	19.2675	22.7331	3.4655	4.0349	0	23.608933	21.642042	25.575825	3.9338	21.3200	0
27-Jul-00	294	27	23.0141	21.4361	24.5920	3.1559	14.3350	0	23.612076	21.645288	25.578864	3.9336	10.1642	0
24-Aug-00	336	20	23.4917	21.6218	25.3616	3.7398	10.1622	0	23.615963	21.649051	25.582875	3.9338	10.9698	0
28-Sep-00	315	27	24.6410	22.8500	26.4319	3.5819	3.7299	0	23.621576	21.654687	25.588466	3.9338	8.7066	0
07-Nov-00	182	26	25.9059	24.9560	26.8558	1.8998	0.1615	1	23.750446	21.788479	25.712412	3.9239	3.0752	1
14-Dec-00	105	25	24.8447	23.1611	26.5283	3.3672	0.3054	1	23.629272	21.662461	25.596083	3.9336	3.1261	1
01-Feb-01	63	22	22.5748	21.3745	23.7750	2.4005	0.7084	1	23.573651	21.607729	25.539573	3.9318	3.3876	1
01-Mar-01	182	23	23.7104	22.4793	24.9416	2.4623	0.7561	1	23.639267	21.673364	25.605169	3.9318	0.6374	1
05-Apr-01	350	26	23.8160	22.2045	25.4275	3.2230	4.7288	0	23.626759	21.660029	25.59349	3.9335	5.5876	0
17-May-01	364	21	21.9516	20.3248	23.5785	3.2537	1.1245	1	23.591712	21.625006	25.558419	3.9334	7.2926	0
28-Jun-01	364	20	21.5827	19.7716	23.3938	3.6222	4.1338	0	23.608374	21.641482	25.575266	3.9338	16.4743	0
06-Sep-01	364	19	19.3492	18.0615	20.6368	2.5753	0.0025	1	23.386878	21.422077	25.351679	3.9296	16.7060	0
27-Nov-01	294	17	16.5379	15.4110	17.6649	2.2539	0.0365	1	23.134161	21.171939	25.096383	3.9244	41.0269	0
						2.9045	**4.3888**	**55.00%**				**3.9304**	**9.5296**	**75.00%**

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